Theory of Computation
Midterm Exam, Fall 2009

1. (30 pts) True or false? Mark ‘O’ for true; ‘×’ for false. Score = max{0, Right − ½Wrong}

(a) If $A' \subseteq A$, then $A$ is nonregular.
False

(b) $\{(james)^n(bond)^n \in \Sigma^* \mid n \geq 0\}$ is a regular language over the English alphabet $\Sigma$.
False

(c) For any regular expression $R$, it is always true that $(R^*R \cup R)^* = R^*$
True

(d) If a regular language is infinite, then every DFA that recognizes it contain cycles (when the DFA drawn as a directed graph).
True

(e) For any two languages $A$ and $B$, if $A \subseteq B$ then $A^n \subseteq B^n$ for all $n \geq 1$. Here, $A^2 = AA$, $A^3 = AAA$, and so on.
True

(f) The language $\{w = xyzy \mid x, y, z \in \{0, 1\}^+\}$ is regular.
True

(g) Let $L_4 = L_1L_2L_3$. If $L_1$ and $L_2$ are regular and $L_3$ is not regular, it is possible that $L_4$ is regular.
True

(h) Let $L_1 = L_2 \cap L_3$. If $L_1$ is context-free, then either $L_2$ or $L_3$ is context-free
False

(i) $\{(ab)^na^nb^n \mid n \geq 0\}$ is not context-free.
True

(j) $a^nb^nc^n = \{a^nb^n \mid n \geq 0\}$ is context-free.
True

(k) $\{xyx^n \mid x \in \{0, 1\}^+, y \in \{0, 1\}^*\}$ is regular.
True

(l) $\{a^kb^m \mid m = k + l\}$ is context-free.
True

(m) $\{x \in \{a, b, c\}^* \mid$ the middle symbol of $x$ is $b$, and $x$ is of odd-length$\}$ is regular.
False ($L \cap a \ast bc \ast = \{a^nb^m \mid n \geq 0\}$)

(n) Deterministic context-free languages are closed under union.
False

(o) Linear context-free languages are closed under union. (Note that a language is linear context-free if it can be generated by a grammar whose production rules are of the form $A \to \alpha$, where $A$ is a nonterminal and $\alpha$ contains at most one nonterminal symbol.)
True

2. (5 pts) Prove that $A = \{w \in \{0, 1\}^* \mid$ the number of 0s in $w$ differs from the number of 1s $\}$ is nonregular. (Hint: you may use any of the closure properties of regular languages.)
Solution: $\overline{A} \cap 0^*1^* = \{0^n1^n \mid n \geq 0\}$

3. (10 pts) Use the pumping lemma to prove that $L = \{x\#y \mid x, y \in \{0, 1\}^*, \text{ when viewed as binary numbers}, x + y = 3y\}$ is nonregular. Example: $1000\#100 \in L$.
Solution: Let $w = 100^k\#10^k$. 

4. (5 pts) Convert the following DFA into an equivalent regular expression using the state elimination method in the order $q_3, q_1, q_2$.

![DFA Diagram](image)

Figure 1: A finite automaton.

![Regular Expression](image)

So our regular expression is:

$\left(00 \cup (1 \cup 0)(0 \cup 1))(0 \cup 11(0 \cup 1))^*10\right)$.

Figure 2: FA $\rightarrow$ Reg. conversion

5. (15 pts) Suppose we define $\text{max}(L) = \{ w \mid w \in L, \forall z \in \Sigma^* (z \neq \epsilon \to wz \notin L) \}$.

(a) (3 pts) What is $\text{max}(L_1L_2)$, where $L_1 = \{ w \in \{a,b\}^* \mid w \text{ contains exactly one } a \}$ and $L_2 = \{ a \}$?
Solution: $L_1L_2$

(b) (6 pts) If $L$ is regular, so is $\text{max}(L)$. True or False? Give a brief yet convincing argument.
Solution: Yes (See 3)

(c) (6 pts) If $\text{max}(L)$ is regular, $L$ must also be regular. True or False? Give a brief yet convincing argument.
Solution: No. Consider $L = \{ a^n \mid n \text{ is prime } \}$; Then $\text{max}(L) = \emptyset$

6. (10 pts) For the deterministic automaton given below, apply the minimization algorithm to compute the equivalence classes of states. Show clearly the computation steps. List the equivalence classes, and apply the quotient construction to derive a minimized automaton. Draw its graph.
Solution: See Figure 5
7. (8 pts) Let $L$ be a language over $\Sigma$. Two words $x, y \in \Sigma^*$ are $L$-equivalent, denoted by $x \equiv_L y$, if and only if for all words $z \in \Sigma^*$ we have $xz \in L \iff yz \in L$. For the language $L = \{0^m1^n0^k1^l \mid m, n, k, l \geq 1\}$, what are the equivalence classes (with respect to $\Sigma^* = \{0, 1\}^*$) induced by $\equiv_L$? (Note that the union of all the equivalence classes is $\Sigma^*$.)

Solution: $C_0 = \{\varepsilon\}$; $C_1 = 0^+; C^2 = 0^+1^+; C_3 = 0^+1^+0^+; C_4 = 0^+1^+0^+1^+; C_5 = \{0, 1\}^* - (C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_4)$

8. (10 pts) Consider the following grammar $G = (\{S, T, X\}, \{0, 1\}, \{P, S\})$ where $P$ is

$S \rightarrow 1S1 \mid T$
$T \rightarrow 1X1 \mid X$
$X \rightarrow 0X0 \mid 1$

(a) (4 pts) What are the first four strings in lexicographic order generated by $G$? (By lexicographic order we mean the order $\varepsilon, 0, 1, 00, 01, 10, 11, 000...$)

Solution: 1, 010, 111, 00100

(b) (6 pts) Show that $G$ is ambiguous.

Solution: The word 111 has more than one parse tree. See Figure 6

9. (7 pts) Give a language $L \subseteq a^*b^*$ such that $L$ can be accepted by DPDA by final state, but cannot be accepted by DPDA by empty stack. Why? Give a formal proof.

Solution: Consider $\{a^m b^n \mid m > n > 0\}$
Solution: With the minimization algorithm we establish that $q_1 \approx q_6 \approx q_8$, $q_2 \approx q_4$ and $q_3 \approx q_5 \approx q_7$. The resulting quotient automaton, presented as a table, is:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rightarrow {q_1, q_6, q_8}$</td>
<td>${q_3, q_5, q_7}$</td>
<td>${q_1, q_6, q_8}$</td>
</tr>
<tr>
<td>${q_3, q_5, q_7}$</td>
<td>${q_1, q_6, q_8}$</td>
<td>${q_2, q_4}$</td>
</tr>
<tr>
<td>${q_2, q_4}$</td>
<td>F</td>
<td>${q_3, q_5, q_7}$</td>
</tr>
</tbody>
</table>

Figure 5: Min FA.

Figure 6: Two different parse trees.