Basic Recursive Function Theory

- The **primitive recursive functions** are defined by starting with some base set of functions and then expanding this set via rules that create new primitive recursive functions from old ones.

- The base functions are:
  1. Zero function: $Z(x) = 0$
  2. Successor function: $S(x) = x + 1$
  3. Projection function: $\pi_i^n(x_1, \cdots, x_n) = x_i$, $1 \leq j \leq n$
  4. Composition: $h(X) = f(g_1(X), \cdots, g_n(X))$
  5. Primitive Recursion:
     $h(X, 0) = f(X)$;
     $h(X, S(n)) = g(X, h(X, n), n)$

Some Primitive Recursive Functions

Example ($add(x, y) = x + y$)

$$add(x, 0) = f(x);$$
$$add(x, S(n)) = g(x, add(x, n), n)$$

where $f(x) = \pi_1(x) = x$; $g(x_1, x_2, x_3) = S(\pi_2(x_1, x_2, x_3)) = S(x_2)$

Example ($mul(x, y) = x \times y$)

$$mul(x, 0) = f(x);$$
$$mul(x, S(n)) = g(x, mul(x, n), n)$$

where $f(x) = 0$;
$g(x_1, x_2, x_3) = add(\pi_1(x_1, x_2, x_3), \pi_2(x_1, x_2, x_3)) = add(x1, x2)$

Bounded Minimization

An n-ary predicate P is primitive recursive if its characteristic function $X_P : N^n \rightarrow \{0, 1\}$ is primitive recursive.

Definition (Bounded minimization)

Let P be an $(n+1)$-ary primitive recursive predicate and $X \in N^n$.
The **bounded minimization** of P is the function
$\mu_y(y \leq k)(X, y) = min\{y | 0 \leq y \leq k, P(X, y)\}$ if the set is not empty;
$= k + 1$ otherwise.

FACT: Bounded minimization of a primitive recursive predicate is primitive recursive.

FACT: All the primitive recursive functions are total; that is, for any primitive recursive function $f : N^k \rightarrow N$, given k numbers $n_1, \cdots, n_k$, the value $f(n_1, \cdots, n_k)$ is well defined.

Partial Recursive Functions

Definition (Minimization)

Let f be a total function from N^{n+1} to N. The **minimization** of f is the function defined as follows: $\mu_y(X, y) = min\{y | f(X, y) = 0\}$

Definition (Partial recursive function)

The class of **partial recursive functions** includes the primitive recursive functions and also functions defined by minimization.

Ackermann's function

Definition (Ackermann's function)

- $A(0, y) = y + 1$
- $A(x, 0) = A(x - 1, 1)$
- $A(x, y + 1) = A(x - 1, A(x, y))$

- $A(0, n) = n + 1$
- $A(1, n) = 2 + (n + 3) - 3$
- $A(2, n) = 2 * (n + 3) - 3$
- $A(3, n) = 2^{n+3} - 3$
- $A(4, n) = 2^{2^{2^{\cdots 2}}} - 3$ (a stack of $n + 3$ terms)

Theorem

Ackermann's function is not primitive recursive.

Computability

Theorem

Turing computable functions $\equiv$ *Partial recursive functions*