Theory of Computation

Spring 2024, Homework # 2 Reference Solutions

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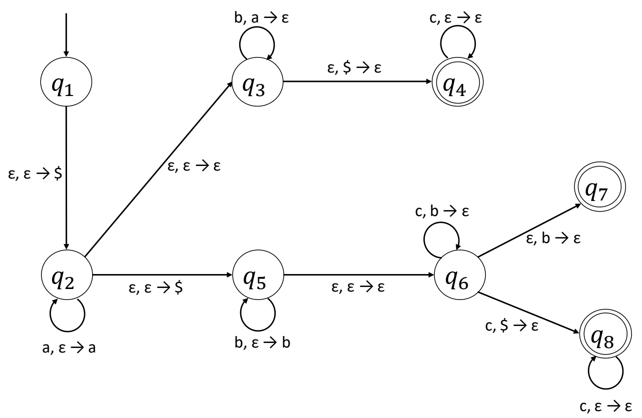
1. (a) Let $L = L_1 \cup L_2$ where $L_1 = \{a^n b^m c^k \mid n = m\}$, $L_2 = \{a^n b^m c^k \mid m \neq k\}$. Let $G = \{V, \Sigma, R, S\}$ be the context-free grammar of L, where R is

   $S \to S_1 \mid S_2$
   $S_1 \to M_1 K$ (S_1 is for L_1.)
   $M_1 \to a\, M_1\, b \mid \epsilon$ (This ensures that $n = m \geq 0$.)
   $K \to c\, K \mid \epsilon$ (This ensures that $k \geq 0$.)
   $S_2 \to N\, M_2$ (S_2 is for L_2.)
   $N \to a\, N \mid \epsilon$ (This ensures that $n \geq 0$.)
   $M_2 \to b\, M_2\, c \mid B \mid C$ (This ensures that either $m > k$ or $m < k$.)
   $B \to b\, B \mid b$ (This ensures that $m \geq 1$.)
   $C \to C\, c \mid c$ (This ensures that $k \geq 1$.)

   (b) The PDA follows acceptance by *Final State*. So it doens't require the stack to be empty upon accepting the input.



2. Assume that C is context-free. Let p be the pumping length and $s = a^p b^p c^p d^p \in C$. Consider a partition $s = uvxyz$ with $|vxy| \leq p$ and $|vy| > 0$. Consider the following cases:

   - If v contains at least one a, then y must not contain any c's (since $|vxy| \leq p$). Therefore $uv^2xy^2z \notin C$ since it has more a's than c's.
   - If v contains at least one b, then similar to the previous case, y must not contain any d's and hence $uv^2xy^2z \notin C$.
   - If v contains only c and/or d, then $uxz \notin C$ since it has fewer than p c's (and/or d's) while the number of a's and b's stay the same (i.e., p).
   - $|v| = 0$. Since $|xy| \leq p$, y cannot contain more than 2 types of symbols. If y contains only one type of symbols, then $uxz \notin C$. If y contains two types of symbols, it still cannot contain a and c (or b and d, respectively) at the same time, so $uxz \notin C$.

So we've shown that pumping lemma doesn't hold for C. Hence C is not context-free.

3. Denote the shuffle operation as $\|$. Let $L_1 = \{a^n c^n \mid n \geq 0\}$, $L_2 = \{b^n d^n \mid n \geq 0\}$, and $L_3 = a^*b^*c^*d^*$. So L_1 and L_2 are CFL's while L_3 is regular. If $L = L_1 \| L_2$ is context-free, then $L_4 = L \cap L_3$ is also context-free since the intersection of a CFL with a regular language is always a CFL (see p. 60 of Chapter 2 lecture notes). However, $L_4 = L \cap L_3 = \{a^n b^m c^n d^m \mid m, n \geq 0\}$ is not context-free (according to Q2). A contradiction. Therefore L is not context-free, which means the class of CFLs is not closed under *shuffle*.

4. Let $P = \{Q, \Sigma, \Gamma, \delta, q_0, F\}$ be a PDA, where
   - $Q = Q_1 \cup Q_2 \cup \{q_0, f\}$, assume $Q_1 \cap Q_2 = \emptyset$ and $q_0, f \notin (Q_1 \cup Q_2)$;
   - $\Gamma = \{\perp, a\}$;
   - $F = \{f\}$;
   - δ is defined as following:

$$\delta(q_0, \epsilon, \epsilon) = (q_{01}, \perp) \tag{1}$$
$$\delta(p, \sigma, \epsilon) = (q, a) \quad \text{if } \delta_1(p, \sigma) = q \tag{2}$$
$$\delta(p, \sigma, a) = (q, \epsilon) \quad \text{if } p \in F_1 \text{ and } \delta_2(q_{02}, \sigma) = q \tag{3}$$
$$\delta(p, \sigma, a) = (q, \epsilon) \quad \text{if } \delta_2(p, \sigma) = q \tag{4}$$
$$\delta(p, \epsilon, \perp) = (f, \epsilon) \quad \text{if } p \in F_2 \tag{5}$$

   First we prove that $w \in L(P) \implies w \in A \lozenge B$. Let $w = w_1 w_2 \in L(P)$, where P executes (3) upon reading the first symbol in w_2. According to the construct of P, $w_1 \in L(M_1) = A$, $w_2 \in L(M_2) = B$, and $|w_1| = |w_2|$. So $L(P) \subseteq A \lozenge B$.
   Then we prove that $w \in A \lozenge B \implies w \in L(P)$. Consider $xy \in A \lozenge B$ where $|x| = |y|$. Since $x \in A$, $\delta_1^*(q_{01}, x) = p_1 \in F_1$. So P excutes (3) upon reading the first symbol in y. (*Note that this is only possible because P is nondeterministic.*) Since $y \in B$, $\delta_2^*(q_{02}, y) = p_2 \in F_2$. So after reading y, P executes (5) and enters f. This means P accepts xy so $A \lozenge B \subseteq L(P)$. This proves that $L(P) = A \lozenge B$, i.e., P accepts $A \lozenge B$.

5. Following are the step-by-step conversion.
   (a) Introduce the new start variable S_0 and a new rule $S_0 \to S$. So the CFG becomes:
   $S_0 \to S$
   $S \to BSB \mid B \mid \epsilon$
   $B \to 00 \mid \epsilon$
   (b) Remove ϵ rules:
   - Removing $B \to \epsilon$. So the CFG becomes:
     $S_0 \to S$
     $S \to BSB \mid BS \mid SB \mid S \mid B \mid \epsilon$
     $B \to 00$
   - Removing $S \to \epsilon$. So the CFG becomes:
     $S_0 \to S \mid \epsilon$
     $S \to BSB \mid BB \mid BS \mid SB \mid S \mid B$
     $B \to 00$

(c) Remove unit rules:

- Removing $S \to S$. So the CFG becomes:
  $S_0 \to S \mid \epsilon$
  $S \to BSB \mid BB \mid BS \mid SB \mid B$
  $B \to 00$
- Removing $S \to B$. So the CFG becomes:
  $S_0 \to S \mid \epsilon$
  $S \to BSB \mid BB \mid BS \mid SB \mid 00$
  $B \to 00$
- Removing $S_0 \to S$. So the CFG becomes:
  $S_0 \to BSB \mid BB \mid BS \mid SB \mid 00 \mid \epsilon$
  $S \to BSB \mid BB \mid BS \mid SB \mid 00$
  $B \to 00$

(d) Replace ill-placed terminals 0 by a new variable U with a new rule $U \to 0$. So the CFG becomes:
$S_0 \to BSB \mid BB \mid BS \mid SB \mid UU \mid \epsilon$
$S \to BSB \mid BB \mid BS \mid SB \mid UU$
$B \to UU$
$U \to 0$

(e) Shorten the rules where RHS contains more than two variables by introducing new variables. So the CFG (the final version) becomes:
$S_0 \to BA_1 \mid BB \mid BS \mid SB \mid UU \mid \epsilon$
$S \to BA_1 \mid BB \mid BS \mid SB \mid UU$
$B \to UU$
$U \to 0$
$A_1 \to SB$

Note that for the final CFG in Chomsky normal form, the start variable is S_0 and the set of variables is $V = \{S_0, S, B, U, A_1\}$.