Supplementary Materials

Finite Automata

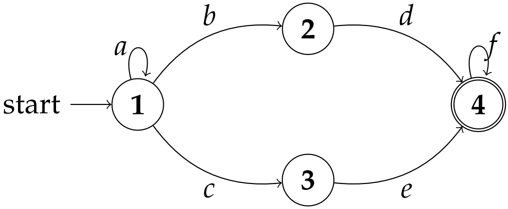


Figure: A Finite Automaton accepting string *abdf*.

Finite Transducers

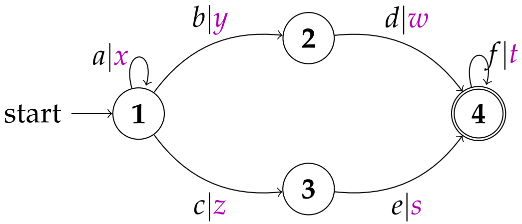


Figure: A Finite Transducer generating string *xywt* on input *abdf*.

Weighted Finite Automata

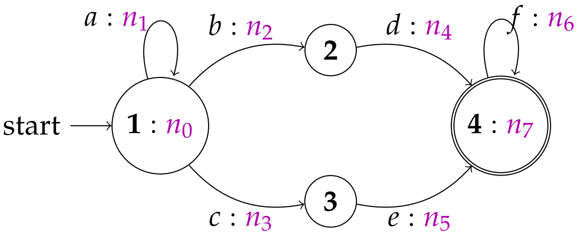


Figure: A Weighted Finite Automaton with weight $n_0 \otimes n_1 \otimes n_2 \otimes n_4 \otimes n_6 \otimes n_7$ on input *abdf*.

Weighted Finite Transducer

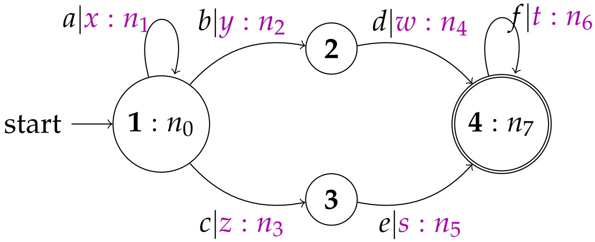
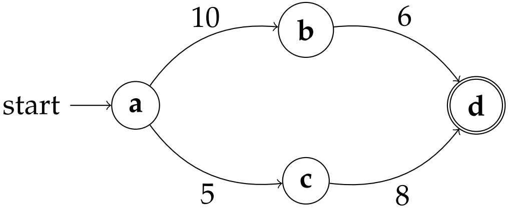


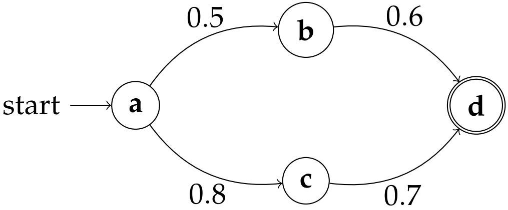
Figure: A Weighted Finite Transducer with output *xywt* and weight $n_0 \otimes n_1 \otimes n_2 \otimes n_4 \otimes n_6 \otimes n_7$ on input *abdf*.

Shortest Path



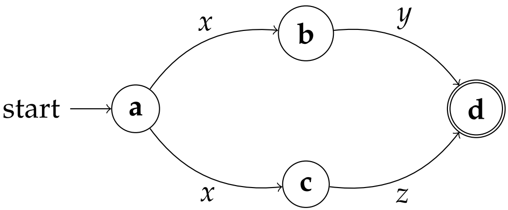
- Compute $10 + 6 = 16$ and $5 + 8 = 13$
- Output $min\{16, 13\}$.

Maximum Reliability



- Compute $0.5 \times 0.6 = 0.3$ and $0.8 \times 0.7 = 0.56$
- Output $max\{0.3, 0.56\}$.

Language Acceptor



- Compute $\{x\} \cdot \{y\}$ and $\{x\} \cdot \{z\} = xz$
- Output $\bigcup\{xy, xz\}$.

Generic Problem Solving

The above three problems were different on the surface, but at the core, they are actually very much the same problem. Consider:

- $min\{(10+6),(5+8)\}$
- $max\{(0.5\times 0.6),(0.8\times 0.7)\}$
- $\bigcup\{\{x\}\cdot\{y\},\{x\}\cdot\{z\}\}$

Hence, it is interesting to see how to unify the above in a single framework – Semiring.

The above three are semirings with operators $(min,+)$, $(max,\times)$ and $(\bigcup,\cdot)$.

Types of "Extended" Finite Automata

Type	Input	Output	Weight	Mapping
Finite Automata (FA)	✓			$\Sigma^* \to \{accept, reject$
Finite Transducer (FT)	✓	✓		$\Sigma^* \to 2^{\Gamma^*}$
Weighted FA (WFA)	✓		✓	$\Sigma^* \to S$
Weighted FT (WFT)	✓	✓	✓	$\Sigma^* \to 2^{\Gamma^*} \times S$

Abstract Algebra – Field

A Field is a 5-tuple $(S, \oplus, \otimes, \overline{0}, \overline{1})$, where S is a set and $\oplus$ and $\otimes$ are two operators, such that

Addition $\oplus$

- Associativity: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
- Commutativity: $a \oplus b = b \oplus a$
- Identity $\overline{0}$: $\overline{0} \oplus a = a \oplus \overline{0} = a$
- Inverse $-a$: $-a \oplus a = a \oplus -a = \overline{0}$

Multiplication $\otimes$

- Associativity: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
- Commutativity: $a \otimes b = b \otimes a$
- Identity $\overline{1}$: $\overline{1} \otimes a = a \otimes \overline{1} = a$
- Inverse a^{-1}: $a^{-1} \otimes a = a \otimes a^{-1} = \overline{1}$

Distributivity of Multiplication over Addition

- $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$
- $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$

Abstract Algebra – Ring

A Ring is a 5-tuple $(S, \oplus, \otimes, \overline{0}, \overline{1})$, where S is a set and $\oplus$ and $\otimes$ are two operators, such that

Addition $\oplus$

- Associativity: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
- Commutativity: $a \oplus b = b \oplus a$
- Identity $\overline{0}$: $\overline{0} \oplus a = a \oplus \overline{0} = a$
- Inverse $-a$: $-a \oplus a = a \oplus -a = \overline{0}$

Multiplication $\otimes$

- Associativity: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
- Identity $\overline{1}$: $\overline{1} \otimes a = a \otimes \overline{1} = a$

Distributivity of Multiplication over Addition

- $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$
- $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$

Example: Square Matrices

A Semiring is a 5-tuple $(S, \oplus, \otimes, \overline{0}, \overline{1})$, where S is a set and $\oplus$ and $\otimes$ are two operators, such that

Addition $\oplus$

- Associativity: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
- Commutativity: $a \oplus b = b \oplus a$
- Identity $\overline{0}$: $\overline{0} \oplus a = a \oplus \overline{0} = a$

Multiplication $\otimes$

- Associativity: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
- Identity $\overline{1}$: $\overline{1} \otimes a = a \otimes \overline{1} = a$

Distributivity of Multiplication over Addition

- $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$
- $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$

Example: Probability

Examples of Semirings

- Probability: $([0,1], +, \times, 0, 1)$
- Boolean: $(\{0,1\}, \vee, \wedge, 0, 1)$
- Tropical: $(R, min, +, \infty, 0)$
- Log : $(R, \oplus_{LOG}, +, \infty, 0)$, where
  $x \oplus_{LOG} y = -\log(e^{-x} + e^{-y})$

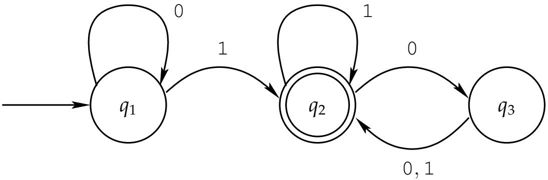


Figure: A Finite Automaton M_1

Consider the following matrix representation:

- Initial state $I = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; final state $F = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$;

$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$; $M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Algebraic View of DFA

The computation $q_1 \xrightarrow{1} q_2 \xrightarrow{0} q_3 \xrightarrow{1} q_2$ is represented by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T$$

As $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \cdot F = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1$, the input "101" is accepted.

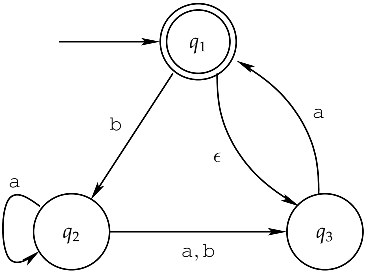


Figure: NFA N_4

$$M_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}; M_b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; M_\epsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

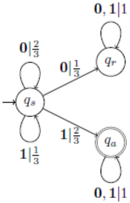
Matrix Multiplication

Question:
How to define matrix multiplication

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}$$ for the above examples"

- In $(a_{1,1} \cdot b_{1,1} + a_{1,2} \cdot b_{2,1} + a_{1,3} \cdot b_{3,1})$, for instance, the operations "." and "+" stand for integer multiplication and addition, resp.
- Suppose "1" and "0" stand for Boolean "True" and "False", resp., the operations "." and "+" stand for Boolean operations $\wedge$ and $\vee$, resp.
- Hence, conventional FA are with respect to $(\vee, \wedge)$-Semiring.

Probabilistic FA: $(+, \times)$-Semiring



PFA A_0: $q_s = q_1, q_r = q_2, q_a = q_3$

$$M_0 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

On input 011, we calculate

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{2}{9} \\ \frac{1}{3} \\ \frac{4}{9} \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{27} \\ \frac{1}{3} \\ \frac{16}{27} \end{pmatrix}^T$$

, where $\frac{16}{27}$ corresponds to

- $q_s \overset{0|\frac{2}{3}}{\rightarrow} a_s \overset{1|\frac{1}{3}}{\rightarrow} q_s \overset{1|\frac{2}{3}}{\rightarrow} q_a \Rightarrow \text{prob.} = \frac{4}{27}$
- $q_s \overset{0|\frac{2}{3}}{\rightarrow} a_s \overset{1|\frac{2}{3}}{\rightarrow} q_a \overset{1|1}{\rightarrow} q_a \Rightarrow \text{prob.} = \frac{4}{9}$

Probabilistic Finite Automaton – Formal Definition

A probabilistic finite automaton (PFA) A is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states;
- Σ is a finite alphabet;
- $\delta : Q \times \Sigma \times Q \to [0, 1]$ is the transition function, such that $\forall q, \in Q, \forall a \in \Sigma, \sum_{q' \in Q} \delta(q, a, q') = 1$, where $\delta(q, a, q')$ is a rational number;
- $q_0 \in Q$ is the start state; and
- $F \subseteq Q$ is the accept states.

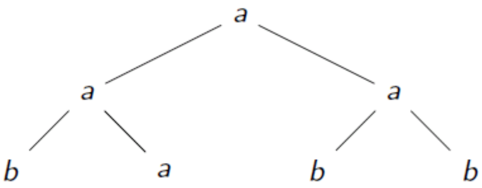
The language $L_{\diamondsuit x}(A) = \{u \in \Sigma^* \mid P_A(u) \diamondsuit x\}$, where $P_A(u)$ is the probability of acceptance on u, $x \in [0, 1]$, and $\diamondsuit \in \{<, \leq, =, \geq, >\}$.

- In general, $L_{\diamondsuit x}(A)$ may not be regular. For instance, $L_{>\frac{1}{2}}(A_0)$ and $L_{\geq\frac{1}{2}}(A_0)$ are not regular.
- $L_{\diamondsuit x}(A)$ is regular, if $x \in \{0, 1\}$.

Why Tree Automata?

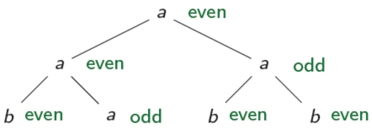
- Foundations of XML type languages (DTD, XML Schema, Relax NG...)
- Provide a general framework for XML type languages
- A tool to define regular tree languages with an operational semantics
- Provide algorithms for efficient validation
- Basic tool for static analysis (proofs, decision procedures in logic)
- ...

E.g. Binary trees with an even number of a's

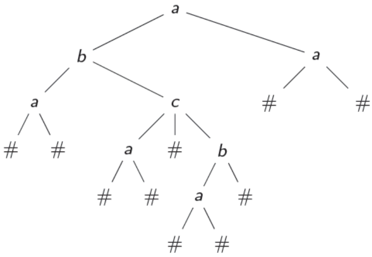


Binary Trees & Ranked Trees

- Binary trees with an even number of a's
- How to write transitions?
  - (even, odd) $\xrightarrow{a}$ even
  - (even, even) $\xrightarrow{a}$ odd
  - ...



- Ranked Tree:
  - Alphabet: $\{a^{(2)}, b^{(2)}, c^{(3)}, \#^{(0)}\}$
  - $a^{(k)}$: symbol a with arity(a) = k



Bottom-up (Ranked) Tree Automata

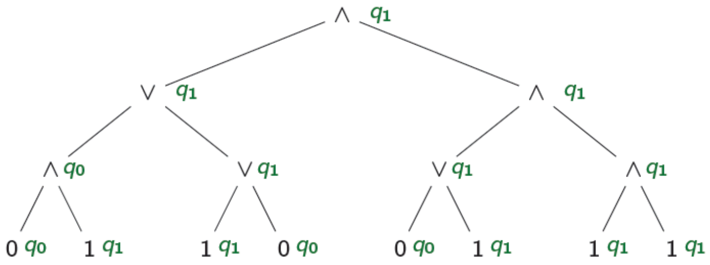
A **ranked bottom-up tree automaton** A consists of:

- $Alphabet(A)$: finite alphabet of symbols
- $States(A)$: finite set of states
- $Rules(A)$: finite set of transition rules
- $Final(A)$: finite set of final states ($\subseteq States(A)$)

where $Rules(A)$ are of the form $(q_1, \ldots, q_k) \xrightarrow{a^{(k)}} q$;

if $k = 0$, we write $\epsilon \xrightarrow{a^{(0)}} q$

Bottom-up Tree Automata: An Example



Principle

- Alphabet$(A) = \{\wedge, \vee, 0, 1\}$
- States$(A) = \{q_0, q_1\}$
- 1 accepting state at the root: Final$(A) = \{q_1\}$

Rules(A)

$\epsilon \xrightarrow{0} q_0$	$\epsilon \xrightarrow{1} q_1$
$(q_1, q_1) \xrightarrow{\wedge} q_1$	$(q_0, q_1) \xrightarrow{\vee} q_1$
$(q_0, q_1) \xrightarrow{\wedge} q_0$	$(q_1, q_0) \xrightarrow{\vee} q_1$
$(q_1, q_0) \xrightarrow{\wedge} q_0$	$(q_1, q_1) \xrightarrow{\vee} q_1$
$(q_0, q_0) \xrightarrow{\wedge} q_0$	$(q_0, q_0) \xrightarrow{\vee} q_0$

Top-down (Ranked) Tree Automata

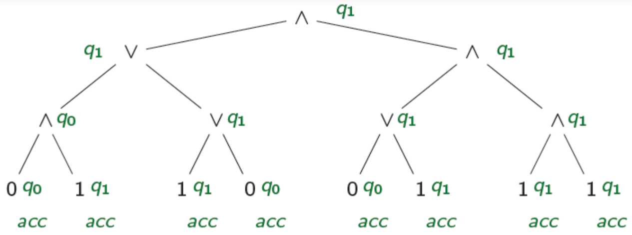
A **ranked top-down tree automaton** A consists of:

- $Alphabet(A)$: finite alphabet of symbols
- $States(A)$: finite set of states
- $Rules(A)$: finite set of transition rules
- $Final(A)$: finite set of final states ($\subseteq States(A)$)

where $Rules(A)$ are of the form $q \xrightarrow{a^{(k)}} (q_1, ..., q_k)$;
if $k = 0$, we write $\epsilon \xrightarrow{a^{(0)}} q$

Top-down tree automata also recognize all regular tree languages

Top-down Tree Automata: An Example



Principle

- starting from the root, guess correct values
- check at leaves
- 3 states: q_0, q_1, acc
- initial state at the root: q_1
- accepting if all leaves labeled acc

Transitions

$q_1 \xrightarrow{\wedge} (q_1, q_1)$ $\quad q_1 \xrightarrow{\vee} (q_0, q_1)$

$q_0 \xrightarrow{\wedge} (q_0, q_1)$ $\quad q_1 \xrightarrow{\vee} (q_1, q_0)$

$q_0 \xrightarrow{\wedge} (q_1, q_0)$ $\quad q_1 \xrightarrow{\vee} (q_1, q_1)$

$q_0 \xrightarrow{\wedge} (q_0, q_0)$ $\quad q_0 \xrightarrow{\vee} (q_0, q_0)$

$q_1 \xrightarrow{1} \text{acc}$ $\quad q_0 \xrightarrow{0} \text{acc}$

Expressive Power of Tree Automata

Theorem 1

The following properties are equivalent for a tree language L:

(a) *L is recognized by a* **bottom-up non-deterministic** *tree automaton*

(b) *L is recognized by a* **bottom-up deterministic tree** *automaton*

(c) *L is recognized by a* **top-down non-deterministic** *tree automaton*

(d) *L is generated by a* **regular** *tree grammar*

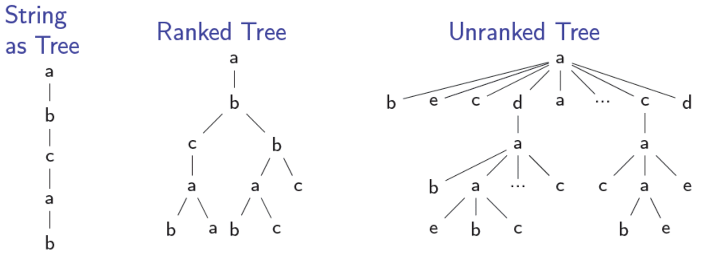
Deterministic Top-down Tree Automata

Deterministic top-down tree automata do not recognize all regular tree languages

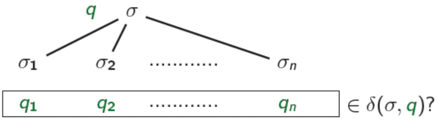
- Example:

$\text{Initial}(A) = q_0$
$q_0 \xrightarrow{a} (q, q)$
$q \xrightarrow{b} \epsilon$
$q \xrightarrow{c} \epsilon$
also accepts...

Unranked Trees



$\delta(\sigma, q)$: specified by a regular expression (i.e., regular language).



Quantum Entanglement

- An n-qubit system can exist in any superposition of the 2^n basis states.

$$\alpha_0|000\ldots000\rangle + \alpha_1|000\ldots001\rangle + \cdots + \alpha_{2^n-1}|111\ldots111\rangle$$

- Sometimes such a state can be decomposed into the states of individual bits

$$\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \frac{1}{\sqrt{2}}((|0\rangle + |1\rangle))$$

- But,

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

is not decomposible, which is called an entangled state.

Unitary Evolution

- A quantum system that is not measured (i.e. does not interact with its environment) evolves in a unitary fashion.
- That is, it's evolution in a time step is given by a unitary linear operation.
- Such an operator is described by a matrix U such that

$$UU^* = I$$

where U^* is the conjugate transpose of U.

$$\begin{pmatrix} 3 & 3+i \\ 2-i & 2 \end{pmatrix}^* = \begin{pmatrix} 3 & 2+i \\ 3-i & 2 \end{pmatrix}$$

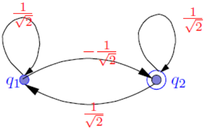
- Quantum finite automata are obtained by letting the matrices M_σ have complex entries. We also require each of the matrices to be unitary. E.g.

$$M_\sigma = \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix}$$

- If all matrices only have 0 or 1 entries and the matrices are unitary, then the automaton is deterministic and reversible.

Quantum Automata

Consider the automaton in a one letter alphabet as:



$$M_a = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

- The initial state $|\psi_0\rangle = 1 \cdot |0\rangle + 0 \cdot |1\rangle = (1, 0)^T$
- $M_{aa} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence, upon reading *aa*, M's state is

  $$|\psi\rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0 \cdot |0\rangle + -1 \cdot |1\rangle$$
- There are two distinct paths labelled *aa* from q_1 back to itself, and each has non-zero probability, the net probability of ending up in q_1 is 0.
- The automaton accepts a string of odd length with probability 0.5 and a string of even length with probability 1 if its length is not a multiple of 4 and probability 0 otherwise.

Measure-once Quantum Automata

- The accept state of the automaton is given by an $N \times N$ projection matrix P, so that, given a N-dimensional quantum state $|\psi\rangle$, the probability of $|\psi\rangle$ being in the accept state is $\langle\psi|P|\psi\rangle = \||P|\psi\rangle\|^2$.
In the previous example, $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
- The probability of the state machine accepting a given finite input string $\sigma = (\sigma_0, \sigma_1, \cdots, \sigma_k)$ is given by
$Pr(\sigma) = \|PU_{\sigma_k} \cdots U_{\sigma_1} U_{\sigma_0}|\psi\rangle\|^2$. In the previous example, Pr(aa)=
$\begin{pmatrix} 0 \\ -1 \end{pmatrix}^T \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 1$
- A regular language is accepted with probability p by a quantum finite automaton, if, for all sentences σ in the language, (and a given, fixed initial state $|\psi\rangle$), one has $p < Pr(\sigma)$.

Language Accepted

- Measure Many 1-way QFA: Measurement is performed after each input symbol is read.
- Measure-many model is more powerful than the measure-once model, where the power of a model refers to the acceptance capability of the corresponding automata.
- MM-1QFA can accept more languages than MO- 1QFA.
- Both of them accept proper subsets of regular languages.