Theory of Computation

Spring 2021, Midterm Exam (Solutions)

1. (12 pts) We showed in class that the DFA-acceptable languages were closed under intersection. The construction that established this is called the $\boxed{\text{.....product automata construction.....}}$. It works liked this. Given a DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and a DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, we construct a DFA $M = (Q, \Sigma, \delta, q_0, F)$ for $L(M_1) \cap L_(M_2)$. For the state set of M, we select $Q = \boxed{Q_1 \times Q_2}$. (Thus if M_1 has 10 states and M_2 has 20 states, then M will have $\boxed{200}$ states.) For the start state of M we set $q_0 = \boxed{(q_1, q_2)}$. The final states of M are set $F = \boxed{F_1 \times F_2}$. The transition function δ is $\boxed{\delta(q, a) = (\delta_1(q, a), \delta_2(q, a))}$. Fill in blanks (1) - (6).

2. (8 pts) Give a CFG for the language $L = \{xx^R yy^R \mid x, y \in \{a, b\}^*\}$. Your CFG must be as simple as possible. Here x^R stands for the reversal of string x, e.g., $(abc)^R = cba$.
Solution:
$S \to S_1 S_2$
$S_1 \to aS_1a \mid bS_1b \mid \epsilon$
$S_2 \to aS_2a \mid bS_2b \mid \epsilon$

3. (12 pts) Let $\Sigma = \{a, b, c, d\}$, and $L_1, L_2 \subseteq \Sigma^*$ with

$$L_1 = \{w \mid w \text{ contains equal number of } a's \text{ and } b's\}$$

$$L_2 = \{w \mid w \text{ contains equal number of } c's \text{ and } d's\}$$

Let $L = L_1 \cap L_2$. Answer the following questions along with brief explanations.

   (a) Are L_1 and L_2 context-free?
   Solution: YES. Here is the PDA that recognizes L_1 (one for L2 is similar): the PDA ignores the symbols c and d. It pushes every a (resp. b) to the stack unless the top symbol of the stack is b (resp. a), in which case, it pops the top symbol. In the end, the PDA accepts if the stack is empty. Note that the PDA is deterministic.

   (b) Are $\overline{L_1}$ and $\overline{L_2}$ context-free?
   Solution: YES. The PDA for $\overline{L_1}$ is the same as the PDA for L_1 described above, except that the Accept/Reject decision is reversed (this makes sense because the PDA is deterministic).

   (c) Is L context-free?
   Solution: NO. Note that $(L_1 \cap L_2) \cap a^*c^*b^*d^* = \{a^n c^m b^n d^m \mid m, n \geq 0\}$.

   (d) Is $\overline{L}$ context-free?
   Solution: YES. $\overline{L} = \overline{L_1} \cup \overline{L_2}$

   Recall that $\overline{L} = \Sigma^* - L$.

4. (8 pts) For $n \in N$, $F(n)$ is the n-th Fibonacci number defined as $F(1) = 1$; $F(2) = 1$; $\forall n > 2, F(n) = F(n-1) + F(n-2)$. For $\Sigma = \{a\}$, consider the language $L = \{a^m \mid m = F(n), n > 0\}$. Is L regular? Justify your answer.
Solution: Not regular. Assume that L were regular. Let n be the pumping constant. We choose a p such that $F(p-1) > n$. Consider $a^{F(p)} \in L$. For every u, v, w with $a^{F(p)} = uvw$ and $|v| \leq n$, $|uv^2w| = F(p) + |v| \leq F(p) + n < F(p) + F(p-1) = F(p+1)$, hence, $wv^2w \notin L$.

5. (10 pts) Consider the DFA given below, in which q_0 is the initial state and q_3 is the final state. Find the equivalent minimum DFA. Show your work in sufficient detail.

		a	b	
$\rightarrow$	q_0	q_0	q_1	
	q_1	q_2	q_3	
	q_2	q_2	q_3	
	q_3	q_2	q_4	F
	q_4	q_0	q_1	

Solution:

		a	b	
$\rightarrow$	$\{q_0, q_4\}$	$\{q_0, q_4\}$	$\{q_1, q_2\}$	
	$\{q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_3\}$	
	$\{q_3\}$	$\{q_1, q_2\}$	$\{q_0, q_4\}$	F

6. (10 pts) Answer the following questions:

   (a) For $L_1, L_2 \subseteq \{a, b\}^*$, let $L_1 \cdot_{\#(a)} L_2 = \{uv \mid u \in L_1, v \in L_2,\ \#_a(u) = \#_a(v)\}$, where $\#_a(u)$ denotes the number of a's in u. For example, $\#_a(baaba) = 3$. Given L_1, L_2 regular, is $L_1 \cdot_{\#(a)} L_2$ always regular? Justify your answer.
   Solution: No. Suppose $L_1 = a^*c$ and $L_2 = a^*$. $L_1 \cdot_{\#(a)} L_2 = \{a^n c a^n \mid n \geq 0\}$ which is not regular.

   (b) Given L_1, L_2 context-free, is shuffle(L_1, L_2) always context-free? Justify your answer.
   Solution: No. Suppose $L_1 = \{a^n b^n \mid n \geq 0\}$ and $L_2 = \{c^n d^n \mid n \geq 0\}$. shuffle$(L_1, L_2) \cap a^*c^*b^*d^* = \{a^n b^m c^n d^m \mid m, n \geq 0\}$, which is not context-free.

7. (10 pts) For any language A, define the set $A_{-*-} = \{y \mid \exists x, z, |x| = |y| = |z|, xyz \in A\}$. For example, if $A = \{\underline{\epsilon}, a, ab, b\underline{a}b, bbab, aa\underline{bb}ab\}$, then $A_{-*-} = \{\epsilon, a, bb\}$.

   (Question): Given a regular language L, is L_{-*-} always regular? Justify your answer.

   To received full credit, if you think the answer is negative, give a counter-example and show the example to be non-regular. If your answer is positive, give a convincing argument (proof).
   Solution: Always regular. Let $M = (Q, \Sigma, \delta, q_0, F)$ be an FA accepting A. We construct the following FA $M' = (Q', \Sigma, q_0', \delta;, F')$ to accept A_{-*-}.

   - $Q' = (Q \times Q \times Q) \cup \{q_0'\}$
   - δ :
     - $(q_1, q_2, q_0, q_1, q_2) \in \delta'(q_0, \epsilon), \forall q_1, q_2 \in Q$,
       where q_1 and q_2 capture the states visited in the $\frac{1}{3}$rd and $\frac{2}{3}$rd positions, respectively, along an accepting computation.
     - $(q_1, q_2, p', q', r') \in \delta'((q_1, q_2, p, q, r), a)$ if $p' \in \delta(p, x), q' \in \delta(q, a), r' \in \delta(r, y)$, for some $x, y \in \Sigma$.
   - $F' = \{(q_1, q_2, q_1, q_2, q_f) \mid q_f \in F\}$.

8. (6 pts) A *star-free* regular expression is a regular expression built up from the atomic expression ϵ, $\emptyset$ and $a \in \Sigma$ using only operations: "$\cdot$" (for concatenation), "$+$" (for union), "$\cap$" (for intersection), and "$\sim$" (for complementation). Note that the "$*$" operation is not allowed in star-free

regular expression. Given a star-free regular expression r, we write $L(r)$ to denote the language expressed by r. Note that $L(\sim r) = \Sigma^* - L(r)$. Suppose $\Sigma = \{a, b, c\}$. Answer the following questions:

(a) (3 pts) Show that $\{a, b, c\}^*$ is star-free.
Solution: $\sim \emptyset = \{a, b, c\}^*$.

(b) (3 pts) Show that $a^+ = \{a, aa, aaa, ...\}$ is star-free.
Solution: $(\sim \epsilon) \cap (\sim (\Sigma^*(b+c)\Sigma^*)$

9. (12 pts) Suppose L is a language over an alphabet Σ. Given a string $x \in \Sigma^*$, we define the suffix language $suf_L(x) = \{y \in \Sigma^* \mid xy \in L\}$. For example, let $\Sigma = \{0, 1\}$ and $L = \{0^n 1^n \mid n \geq 0\}$, then $suf_L(0) = \{1, 011, 00111...\}$ and $suf_L(00) = \{0111, 001111, ...\}$.

(Question): Suppose $\Sigma = \{a, b\}$ and $L = \{x \in \Sigma^* \mid x \text{ contains } ab \text{ as a substring}\}$. For example, $aaabaaa \in L, bbbaaa \notin L$.

(a) (4 pts) It is not hard to see that $suf_L(\epsilon) = L$. What are the sets $suf_L(a)$, $suf_L(b)$, $suf_L(aa)$, $suf_L(ab)$, $suf_L(ba)$, $suf_L(bb)$?
Solution: $suf_L(a) = b\Sigma^* \cup L$; $suf_L(b) = L$; $suf_L(aa) = b\Sigma^* \cup L$, $suf_L(ab) = \Sigma^*$, $suf_L(ba) = b\Sigma^* \cup L$, $suf_L(bb) = L$

(b) (4 pts) What is the size of the set $\{suf_L(x) \mid x \in \Sigma^*\}$? That is, what is the number of different $suf_L(x), \forall x \in \Sigma^*$? Briefly explain why.
Solution: 3, corresponding to Σ^*, L, and $b\Sigma^* \cup L$. For $suf_L(x)$, we have the following cases:

- $x = ya, y \in \Sigma^*$, $suf_L(x) = b\Sigma^* \cup L$,
- $x = yab, y \in \Sigma^*$, $suf_L(x) = \Sigma^*$
- otherwise, $suf_L(x) = L$.

(c) (4 pts) Suppose we want to construct a DFA M to accept L in such a way that each state of M corresponds to a set $suf_L(x)$, where $x \in \Sigma^*$. In M, what is $\delta(suf_L(x), a)$? That is, what is the state resulting from reading a in state $suf_L(x)$? Why?
Solution: $\delta(suf_L(x), a) = suf_L(xa)$

10. (12 pts) Given languages L_1, L_2, recall that $L_1 L_2 = \{uv \mid u \in L_1, v \in L_2\}$. We also define $L_1^{-1} L_2 = \{v \mid \exists u \in L_1, uv \in L_2\}$. Decide whether each of the following equations is true or false. Justify your answers.

(a) $\{a\}^{-1}(\{a\}L) = L$
Solution: True: $x \in \{a\}^{-1}(\{a\}L) \Leftrightarrow ax \in aL \Leftrightarrow x \in L$

(b) $\{a\}(\{a\}^{-1}L) = L$
Solution: False: if $L = \{\epsilon\}$, then $\{a\}^{-1}L = \emptyset$

(c) $L_1^{-1}(L_1 L_2) = L_2$
Solution: False: Take $L_1 = \{a, ab\}, L_2 = \{ba\}$, then $L_1 L_2 = \{aba, abba\}$ and $L_1^{-1}(L_1 L_2) = \{ba, bba, a\}$

(d) $L^{-1}L = \{\epsilon\}$
Solution: False $L = \{a, aa\}$, then $L^{-1}L = \{\epsilon, a\}$