Theory of Computation

Midterm Exam, Nov. 14, 2011

1. (21 pts) Are the following statements true or false. Justify by a proof or a suitable counterexample.

   (a) If L_1 is finite and $L_1 \cup L_2$ is regular then L_2 is regular.
   True. Let $L_3 = (L1 \cup L2) \cap (\bar{L_1})$ - this is regular. Therefore $L_2 = L_3 \cup (L_1 \cap L_2)$ is regular as the second component is finite (and regular).

   (b) If L_1 is regular (and infinite) and $L_1 \cdot L_2$ is regular then L_2 is regular.
   False. Consider $L_1 = 0^*$ and $L_2 = 0^p$ where p is prime, then $L_1 \cdot L_2 = 000^*$ which is regular.

   (c) If L^* is regular, so is L.
   False. Consider $L_2 = 0^p$ where p is prime, then $L_2^* = 000^*$ which is regular.

   (d) Consider a language L and a homomorphism h. If $h(L)$ is regular, then L is always regular.
   False. Take $\{0^n 1^n \mid n \geq 0\}$ and $h(0) = 0$ and $h(1) = \epsilon$.

   (e) Let $L_1 \subseteq L_2$ (over alphabet Σ) both be regular languages. If L_2 can be accepted by a DFA with n states, then L_1 can always be accepted by some DFA with no more than n states.
   False. Consider $L_2 = \Sigma^*$, which can be accepted by a DFA with one state.

   (f) $(R \cup S)^* = R^* \cup S^*$, where R and S are two languages.
   False. Let $R = \{a\}$ and $S = \{b\}$. Then string $ab \in (R \cup S)^*$ but $ab \notin R^* \cup S^*$.

   (g) $(R \cap S)T = RT \cap ST$, where R, S and T are languages.
   False. Let $R = \{\epsilon\}$ and $S = \{a\}$ and $T = a^*$. Then string $a \in RT \cap ST$ but $a \notin (R \cap S)T$.

2. (4 pts) Give a regular expression for the language containing all strings of 0's and 1's such that every pair of adjacent 0's appears before any pair of adjacent 1's. (For example, 01<u>00</u>1011011 is in the language, while 011<u>00</u>1011 is not.)
   Sol.: $((1 + (0 + 01)^*)00)^*((1 + 01)^* + 0)$

3. (10 pts) A *shuffle* of two strings $x, y \in \Sigma^*$ denoted by $x||y$ is the set of strings that can be obtained by interleaving the strings x and y in any manner. For example $ab||cd = \{abcd, acbd, acdb, cabd, cadb, cdab\}$. (The strings need not be of the same length.) For two sets of strings A, B, the shuffle is defined as $A||B = \bigcup_{x \in A, y \in B} x||y$. Prove that if both A and B are regular, then $A||B$ is also regular.
   Sol. Let $M_A = (Q_A, \Sigma, \delta_A, q_{A,0}, F_A)$ and $M_B = (Q_B, \Sigma, \delta_B, q_{B,0}, F_B)$ be FA accepting A and B, respectively. Construct $M = (Q, \Sigma, \delta, q_0, F)$ to accept $A||B$ as follows.

   - $Q = Q_A \times Q_B$
   - $q_0 = (q_{A,0}, q_{B,0})$
   - $F = F_A \times F_B$
   - – If $q'_A \in \delta_A(q_A, a)$, then $(q'_A, r_B) \in \delta((q_A, r_B), a)\ \forall r_B \in Q_B$

– If $q_B' \in \delta_B(q_B, a)$, then $(r_A, q_B') \in \delta((r_A, q_B'), a)\ \forall r_A \in Q_A$

4. (10 pts) Define $L_1\#L_2 = \{x\#y \mid x \in L_1, y \in L_2, |x| = |y|\}$, where $\#$ is a new symbol. Is the following statement true or false? Justify your answer.
   — If L_1 and L_2 are regular, then $L_1\#L_2$ is also regular.
   False: Consider $L_1 = 0^*$ and $L_2 = 1^*$. $L_1\#L_2 = \{0^n\#1^n \mid n \geq 0\}$ – which is not regular.

5. (10 pts) Use the pumping lemma to show that $L = \{0^n1^m \mid n, m \geq 1$ and m leaves a remainder of 3 when divided by $n\}$ is not regular. (For example, 0^41^7, 0^51^{13} are in L.)
   (Hint: Let p be the pumping constant. Take $w = 0^{p+4}1^{p+7}$.)

   Solution: Observe that if $0^n1^m \in L$ then $n > 3$, as otherwise the remainder can never be 3.

   Let $p \geq 0$ be the pumping length. Take, $w = 0^{p+4}1^{p+7}$; clearly, $w \in L$ as $p+4 > 3$ no matter what p is, and $p+7$ leaves a remainder of 3 when divided by $p+4$. Let x, y, z be any partition of w such that $w = xyz$, $|xy| \leq p$ and $|y| > 0$.

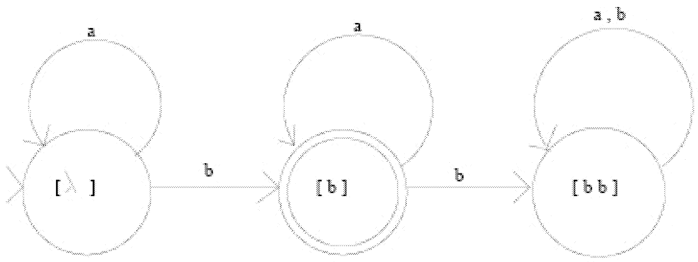
   Since $|xy| \leq p$, we can conclude (without loss of generality) that $x = 0^r$, $y = 0^s$ and $z = 0^t1^{p+7}$, where $r+s+t = p+4$. Further, since $|y| > 0$, we have $s > 0$. Now, $xy^2z = 0^{r+2s+t}1^{p+7} = 0^{(p+4)+s}1^{p+7}$. Depending on whether $p+4+s \leq p+7$ or $p+4+s > p+7$, we have $(p+7) \bmod (p+4+s)$ is either ≤ 2 or $p+7 > 3$. Thus, $xy^2z \notin L$, and L does not satisfy the pumping lemma. Therefore, L is not regular. $\square$

6. (10 pts) Given a language $L \subseteq \Sigma^*$ and two strings $x, y \in \Sigma^*$, $x \equiv_L y$ iff $\forall z \in \Sigma^*$, $xz \in L \Leftrightarrow yz \in L$. Give the $\equiv_L$ equivalence classes of the language $L = a^*ba^*$. Also draw a minimum DFA accepting L.

   Solution: Here are the equivalence classes:

$$[\lambda]_{\equiv_L} = \boldsymbol{a}^*,\ [b]_{\equiv_L} = \boldsymbol{a}^*\boldsymbol{b}\boldsymbol{a}^*,\ [bb]_{\equiv_L} = (\boldsymbol{a} \cup \boldsymbol{b})^*\boldsymbol{b}(\boldsymbol{a} \cup \boldsymbol{b})^*\boldsymbol{b}(\boldsymbol{a} \cup \boldsymbol{b})^*,$$

   and here is the minimal state DFA M_L:



7. (10 pts) Let L be a language. Show that if every subset of L is regular, then L must be finite (i.e., containing a finite number of strings).
   (Hint: Prove it by contradiction. Note that for every $w \in L$, there exists a $w' \in L$ such that $|w'| > 2|w|$. ($|w|$ denotes the length of w.) Use the pumping lemma if needed.)

Proof. We will prove it by contradiction. Assume that L were infinite. Let w_0 be an arbitrary string in L. Let $w_i \in L$ with $|w_i| > 2|w_{i-1}|$, where $i = 1, 2, \ldots$. Since L is infinite, such strings w_i's exist. Let $L_0 = \{w_0, w_1, \ldots, \}$. Then L_0 is a subset of L, and L_0 is infinite. We note that L_0 so constructed has the following property: For any two strings $u \in L_0$ and $v \in L_0$, if v is longer than u, then

$$|v| > 2|u|. \tag{1}$$

By assumption, L_0 is regular. It follows from the pumping lemma that there exists a positive integer K such that for any $w \in L_0$ with $|w| \geq K$, there must be strings x, y, z such that $w = xyz$, $y \neq \epsilon$, and $xy^2z \in L_0$. But $|xy^2z| = |w| + |y| \leq 2|w|$, and so Inequality 1 is violated, which implies that $xy^2z \notin L_0$. This is a contradiction. Thus, L must be finite. This completes the proof.

8. (10 pts) Consider the ϵ-NFA defined in Figure 1 (where $\rightarrow$ and $*$ mark the initial and final states, respectively):

	ϵ	a	b	c
$\rightarrow p$	ϕ	$\{p\}$	$\{q\}$	$\{r\}$
q	$\{p\}$	$\{q\}$	$\{r\}$	ϕ
$*r$	$\{q\}$	$\{r\}$	ϕ	$\{p\}$

Figure 1: An ϵ-NFA.

(a) (4 pts) Compute the ϵ-closure of each state.

(b) (6 pts) Convert the automaton to a DFA.

Solution:

$$\begin{aligned} \epsilon - closure(p) &= \{p\} \\ \epsilon - closure(q) &= \{p, q\} \\ \epsilon - closure(r) &= \{p, q, r\} \end{aligned}$$

Solution: □

	a	b	c
$\rightarrow \{p\}$	$\{p\}$	$\{p, q\}$	$\{p, q, r\}$
$\{p, q\}$	$\{p, q\}$	$\{p, q, r\}$	$\{p, q, r\}$
$*\{p, q, r\}$	$\{p, q, r\}$	$\{p, q, r\}$	$\{p, q, r\}$

9. (15 pts) Consider the DFA given in Figure 2. Suppose we want to find an equivalent minimum DFA.

(a) (10 pts) Use the table filling method discussed in class to find all distinguishable pairs of states. Show $T[i, j]$ for $1 \leq i < j \leq 4$. Mark $T[i, j]$ with an "X" if there exists a string w that can tell i and j apart as far as reaching a final state is concerned. Show your work in sufficient detail.

(b) (5 pts) Draw the minimum DFA.

Answer:

T	1	2	3	4
1		X	X	X
2			X	X
3				
4				

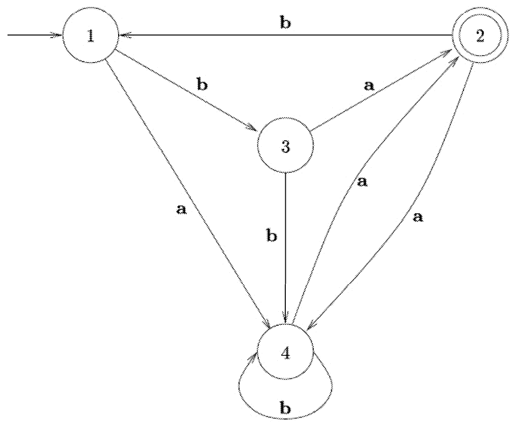


Figure 2: A DFA.

Answer: