Theory of Computation

Midterm Exam, Fall 2009

1. (30 pts) True or false? Mark 'O' for true; '×' for false. Score=max$\{0, Right - \frac{1}{2}Wrong\}$

   (a) If A' is a nonregular language and $A' \subseteq A$, then A is nonregular.
   False

   (b) $\{(james)^n(bond)^n \in \Sigma^* \mid n \geq 0\}$ is a regular language over the English alphabet Σ.
   False

   (c) For any regular expression R, is it always true that $(R^*R \cup R)^* = R^*$
   True

   (d) If a regular language is infinite, then every DFA that recognizes it contain cycles (when the DFA drawn as a directed graph).
   True

   (e) For any two languages A and B, if $A \subseteq B$ then $A^n \subseteq B^n$ for all $n \geq 1$. Here, $A^2 = AA$, $A^3 = AAA$, and so on.
   True

   (f) The language $\{w = xyzy \mid x, y, z \in \{0,1\}^+\}$ is regular.
   True

   (g) Let $L_4 = L_1L_2L_3$. If L_1 and L_2 are regular and L_3 is not regular, it is possible that L_4 is regular.
   True

   (h) Let $L_1 = L_2 \cap L_3$. If L_1 is context-free, then either L_2 or L_3 is context-free.
   False

   (i) $\{(ab)^n a^n b^n \mid n > 0\}$ is not context-free.
   True

   (j) $a^*b^*c^* - \{a^n b^n c^n \mid n \geq 0\}$ is context-free.
   True

   (k) $\{xyx^R \mid x \in \{0,1\}^+, y \in \{0,1\}^*\}$ is regular.
   True

   (l) $\{a^k b^l a^m \mid m = k + l\}$ is context-free.
   True

   (m) $\{x \in \{a,b,c\}^* \mid$ the middle symbol of x is b, and x is of odd-length$\}$ is regular.
   False ($L \cap a * bc* = \{a^n bc^n \mid n \geq 0\}$)

   (n) Deterministic context-free languages are closed under union.
   False

   (o) Linear context-free languages are closed under union. (Note that a language is *linear context-free* if it can be generated by a grammar whose production rules are of the form $A \rightarrow \alpha$, where A is a nonterminal and α contains at most one nonterminal symbol.)
   True

2. (5 pts) Prove that $A = \{w \in \{0,1\}^* \mid$ the number of 0s in w differs from the number of 1s $\}$ is nonregular. (Hint: you many use any of the closure properties of regular languages.)
   Solution: $\bar{A} \cap 0^*1^* = \{0^n 1^n | n \geq 0\}$

3. (10 pts) Use the pumping lemma to prove that $L = \{x\#y \mid x, y \in \{0,1\}^*,$ when viewed as binary numbers, $x + y = 3y\}$ is nonregular. Example: $1000\#100 \in L$.
   Solution: Let $w = 100^k\#10^k$.

4. (5 pts) Convert the following DFA into an equivalent regular expression using the state elimination method in the order q_3, q_1, q_2.

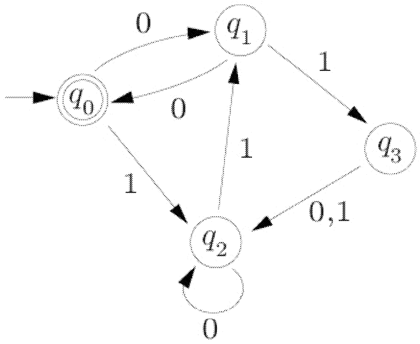

Figure 1: A finite automaton.

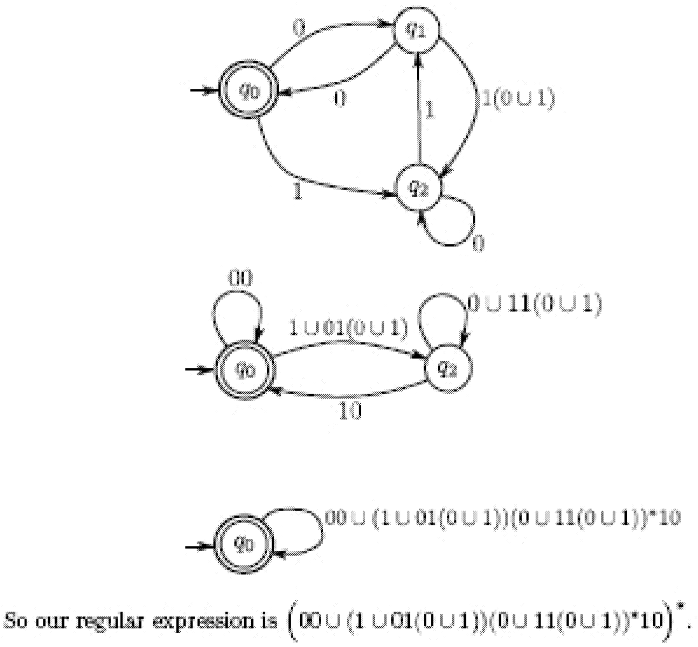

Figure 2: FA → Reg. conversion

5. (15 pts) Suppose we define $max(L) = \{w \mid w \in L, \forall z \in \Sigma^*(z \neq \epsilon \rightarrow wz \notin L)\}$.

   (a) (3 pts) What is $max(L_1L_2)$, where $L_1 = \{w \in \{a,b\}^* \mid w \text{ contains exactly one } a\}$ and $L_2 = \{a\}$?
   Solution: L_1L_2

   (b) (6 pts) If L is regular, so is $max(L)$. True or False? Give a brief yet convincing argument.
   Solution: Yes (See 3)

   (c) (6 pts) If $max(L)$ is regular, L must also be regular. True or False? Give a brief yet convincing argument.
   Solution: No. Consider $L = \{a^n \mid n \text{ is prime }\}$; Then $max(L) = \emptyset$

6. (10 pts) For the deterministic automaton given below, apply the minimization algorithm to compute the equivalence classes of states. Show clearly the computation steps. List the equivalence classes, and apply the quotient construction to derive a minimized automaton. Draw its graph.
Solution: See Figure 5

The proof is by construction. If L is regular, then it is accepted by some DFSA $M = (K, \Sigma, \Delta, s, A)$. We construct a new DFSM $M^* = (K^*, \Sigma^*, \Delta^*, s^*, A^*)$, such that $L(M^*) = maxstring(L)$. The idea is that M^* will operate exactly as M would have except that A^* will include only states that are accepting states in M and from which there exists no path of at least one character to any accepting state (back to itself or to any other). So an algorithm to construct M^* is:

1. Initially, let $M^* = M$.

/* Check each accepting state in M to see whether there are paths from it to some accepting state.

2. For each state q in A do:
   - 2.1. Follow all paths out of q for $|K|$ steps or until the path reaches an element of A or some state it has already visited.
   - 2.2. If the path reached an element of A, then q is not an element of A^*.
   - 2.3. If the path ended without reaching an element of A, then q is an element of A^*.

Figure 3: Regularity proof of $Max(L)$

	a	b
$\rightarrow$ q_1	q_3	q_8
q_2 F	q_3	q_1
q_3	q_8	q_2
q_4 F	q_5	q_6
q_5	q_6	q_2
q_6	q_7	q_8
q_7	q_6	q_4
q_8	q_5	q_8

Figure 4: A deterministic finite automaton.

7. (8 pts) Let L be a language over Σ. Two words $x, y \in \Sigma^*$ are L-equivalent, denoted by $x \equiv_L y$, if and only if for all words $z \in \Sigma^*$ we have $xz \in L \Leftrightarrow yz \in L$. For the language $L = \{0^m 1^n 0^k 1^l \in \{0,1\}^* \mid m, n, k, l \geq 1\}$, what are the equivalence classes (with respect to $\Sigma^* = \{0,1\}^*$) induced by $\equiv_L$? (Note that the union of all the equivalence classes is Σ^*.)
   Solution: $C_0 = \{\epsilon\}$; $C_1 = 0^+$; $C^2 = 0^+1^+$; $C_3 = 0^+1^+0^+$; $C_4 = 0^+1^+0^+1^+$; $C_5 = \{0,1\}^* - (C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_4)$

8. (10 pts) Consider the following grammar $G = (\{S, T, X\}, \{0,1\}, P, S)$ where P is
   $S \rightarrow 1S1 \mid T$
   $T \rightarrow 1X1 \mid X$
   $X \rightarrow 0X0 \mid 1$

   (a) (4 pts) What are the first four strings in lexicographic order generated by G? (By lexicographic order we mean the order $\epsilon, 0, 1, 00, 01, 10, 11, 000 ...$)
   Solution: 1, 010, 111, 00100

   (b) (6 pts) Show that G is ambiguous.
   Solution: The word 111 has more than one parse tree. See Figure 6

9. (7 pts) Give a language $L \subseteq a^*b^*$ such that L can be accepted by DPDA by *final state*, but cannot be accepted by DPDA by *empty stack*. Why? Give a formal proof.
   Solution: Consider $\{a^m b^n \mid m > n > 0\}$

Solution: With the minimization algorithm we establish that $q_1 \approx q_6 \approx q_8$, $q_2 \approx q_4$ and $q_3 \approx q_5 \approx q_7$. The resulting quotient automaton, presented as a table, is:

	a	b
$\to \{q_1, q_6, q_8\}$	$\{q_3, q_5, q_7\}$	$\{q_1, q_6, q_8\}$
$\{q_3, q_5, q_7\}$	$\{q_1, q_6, q_8\}$	$\{q_2, q_4\}$
$\{q_2, q_4\}$ F	$\{q_3, q_5, q_7\}$	$\{q_1, q_6, q_8\}$

Figure 5: Min FA.

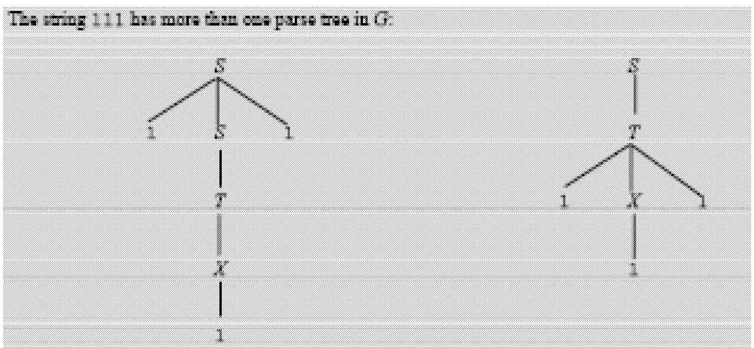

Figure 6: Two different parse trees.