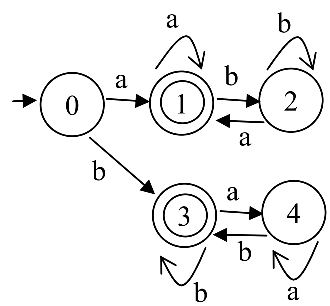
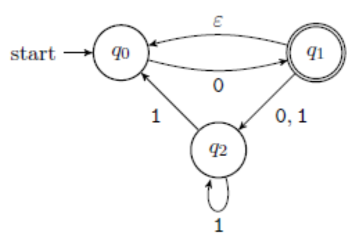
Theory of Computation

Spring 2023, Midterm Exam. (April 18, 2023)

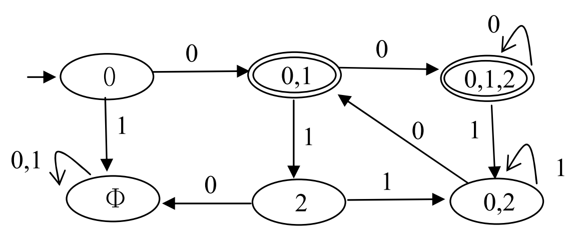
(1) (10 pts) Design a DFA to accept the language $L = \{w \in \{a,b\}^* \mid w \text{ starts and ends with the same symbol}\}$. Note that $\epsilon \notin L$, but $a, b \in L$. Draw the DFA.
Sol.



(2) (10 pts) Convert the following NFA to an equivalent DFA using the subset construction



Sol.



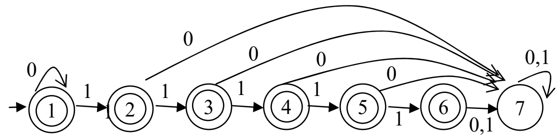
(3) (15 pts) Consider language $L = \{0^n 1^k \in \{0,1\}^* \mid n \geq 0 \wedge k \leq 5\}$. Answer the following questions. You do not need to justify your work.

(a) (10 pts) Give the equivalence classes induced by the $\equiv_L$ relation. (Recall that $x \equiv_L y \Leftrightarrow (\forall z \in \Sigma^*, xz \in L \Leftrightarrow yz \in L)$.)
Sol.

- L is the language represented by a regular expression:

$$0^*(\epsilon + 1 + 11 + 111 + 1111 + 11111)$$

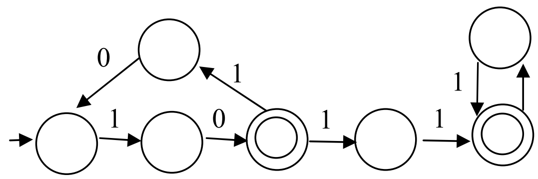
- $C_1 = L(0^*)$, $C_2 = L(0^*1)$, $C_3 = L(0^*11)$, $C_4 = L(0^*111)$, $C_5 = L(0^*1111)$, $C_6 = L(0^*11111)$, $C_7 = (\{0,1\}^* - \bigcup_{i=1}^{6} C_i$.



(b) (5 pts) Draw the minimum DFA that accepts L.
Sol.

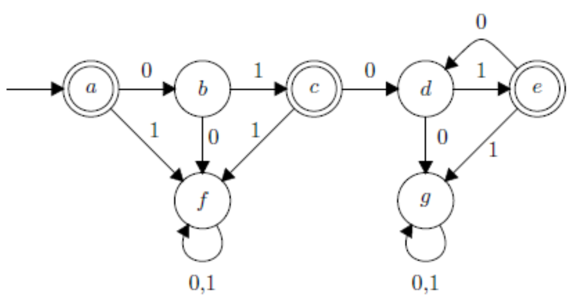
(4) (10 pts)

(a) (5 pts) Draw an NFA for the language $L = \{(10)^n 1^m \mid n \geq 1 \text{ is odd and } m \geq 0 \text{ is even}\}$.
Sol.



(b) (5 pts) Give a context-free grammar for the language, $L = \{0^n 1^n 2^m 3^m \mid n \geq 1 \wedge m \geq 1\}$.
Sol. $S \to AB$; $A \to 0A1 \mid 01$; $B \to 2B3 \mid 23$;

(5) (10 pts) Construct a minimum DFA for the following DFA using the Table Filling algorithm discussed in class. Be sure to show the table as part of your answer.



Sol.

	a	b	c	d	e	f	g
a	-	-	-	-	-	-	-
b	×	-	-	-	-	-	-
c		×	-	-	-	-	-
d	×		×	-	-	-	-
e		×		×	-	-	-
f	×	×	×	×	×	-	-
g	×	×	×	×	×		-

Hence the min DFA has the following transition function

	0	1
$\{a, c, e\}$	$\{b, d\}$	$\{f, g\}$
$\{b, d\}$	$\{f, g\}$	$\{a, c, e\}$
$\{f, g\}$	$\{f, g\}$	$\{f, g\}$

The initial state is $\{a, c, e\}$, which is also the final state.

(6) (10 pts) Give a *right-linear* CFG for the language $\{a^n \mid n \text{ mod } 3 = 2\}$, i.e., the remainder of n divided by 3 is 2). A CFG is *right-linear* if its productions are of the forms either $A \to a \mid \epsilon$, or $A \to bB$.
Sol. $S_0 \to aS_1$; $S_1 \to aS_2$; $S_2 \to aS_0 \mid \epsilon$. The start symbol is S_0.

(7) (10 pts) Consider two possible ways to encode signals: *level transitions* vs. *pulses*. When using pulses, you normally send a 0 and when you want to send a signal, you send a 1. For example. to send signals at times 3, 6, 7, you send string $00\underline{1}00\underline{11}000....$ The same signal can be sent using level transitions by the string $00\underline{1}11\underline{01}111...$, i.e., at times 3, 6, 7 you change one symbol to the other. Suppose we let T be the translation function from pulses to levels. For example, $T(011111) = (010101)$ and $T(111) = 101$. Given a language $L \subseteq \{0,1\}^*$, we define $T(L) = \{T(w) \mid w \in L\}$.
Question: Prove that if L is regular, $T(L)$ is also regular. (Hint: Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA accepting L. Build a DFA $M' = (Q', \Sigma, \delta'.q_0', F')$ to accept $T(L)$. You may consider $Q' = Q \times \{0,1\}$.)
Sol. Construct the following DFA M'

- $Q' = Q \times \{0,1\}$
- $q_0' = (q_0, 0)$
- δ' is as follows:
  - $\delta'((q,0),0) = (\delta(q,0),0)$
  - $\delta'((q,0),1) = (\delta(q,1),1)$
  - $\delta'((q,1),0) = (\delta(q,1),0)$
  - $\delta'((q,1),1) = (\delta(q,0),1)$
- $F' = \{(q,x) \mid q \in F\}$

Note: In M' state $(q,1)$ (resp., $(q,0)$) means the signal is at the "high" (resp., "low") level, and M's state is q. If at a high (resp., low) level and M' sees a 0 (resp., 1), which means a pulse is present. Then the next state of M is $\delta(q,1)$, and M''s level becomes low (resp., high).

(8) (10 pts) Use the pumping lemma to show that $\{0^{2^n} \mid n \geq 0\}$ ($\subseteq 0^*$) is not regular.
Sol. Assume that the language were regular. Let p be the pumping constant. Consider $0^{2^p} = uvw$. For the string uv^2w, $|uv^2w| = 2^p + |v|$. As $|v| \leq p < 2^p$, $2^p + |v| < 2^{p+1}$. Hence, uv^2w is not in the language.

(9) (15 pts) Consider the following context-free grammar: $S \to aSb \mid aSbb \mid ab \mid abb$

(a) (5 pts) Is the grammar ambiguous? Justify your answer.
Sol. Yes. $aabbb$ can be derived by two derivation trees:

i. $S \overset{rule1}{\to} aSb \overset{rule4}{\to} aabbb$
ii. $S \overset{rule2}{\to} aSbb \overset{rule3}{\to} aabbb$

(b) (10 pts) Convert the grammar to an equivalent one in Chomsky Normal Form. Show your work in sufficient detail.
Sol.

$$
\begin{array}{ll}
S \to AP \mid AQ \mid AB \mid AC & \\
P \to SB & Q \to SC \\
A \to a & B \to b \\
C \to BB &
\end{array}
$$

i. Introduce $A \to a$; $B \to b$
ii. Replace $S \to aSb$ by $S \to AP$; $P \to SB$
iii. Replace $S \to aSbb$ by $S \to AQ$; $Q \to SC$; $C \to BB$
iv. Replace $S \to ab$ by $S \to AB$
v. Replace $S \to abb$ by $S \to AC$ (Note that $C \to BB$ has already been introduced)