Theory of Computation

Fall 2015, Final Exam.

Due: January 11, 2015

1. (26 pts) True or False? Score = max{0, Right - $\frac{1}{2}$ Wrong}. No explanations are needed.

   (1) $\{a^m b^n c^k d^l \mid 2m + 5n = 3k + 4l\}$ is context-free.
   True

   (2) $\{a^i b^j c^k \mid i, j, k \geq 0, \text{ and } j > max(i, k)\}$ is not context free.
   True

   (3) If L and L' are deterministic context-free, then $\overline{L \cap L'}$ is context-free.
   True

   (4) $L_k = \{\langle M \rangle \mid M$ halts after at most k steps on $\epsilon\}$ is not decidable.
   False

   (5) If $L \leq_m \{0^n 1^n \mid n \geq 0\}$ then L is recursive. Here $\leq_m$ denotes the many-one reduction.
   True

   (6) If $L \cdot L'$ (i.e., the concatenation of L and L') is not r.e., then either L or L' is not r.e.
   True

   (7) if L_1 and L_2 are in NP, then $L_1 \cup L_2$ is also in NP.
   True

   (8) If L is in P, so is L^*.
   True

   (9) $\{\langle G, w \rangle \mid G$ is a context-free grammar and $w \in L(G)\}$ is in P.
   True

   (10) If both L and L' are in P, $L - L'$ is also in P.
   True

   (11) $\{\langle M \rangle \mid L(M) = \emptyset$, where M is a TM$\}$ is NP-hard.
   True

   (12) Every total function is a partial recursive function.
   False

   (13) $\{\langle M, x \rangle \mid$ on input x, TM M enters some state more than 50 times$\}$ is recursive.
   True

2. (20 pts) Consider the following classes of languages numbered (1)-(6)
   (1) Finite (2) Regular (3) Context-free (4) Recursive (5) Recursively enumerable (r.e.) (6) All languages
   For each of the following languages L, decide an $i, 1 \leq i \leq 6$ such that L is in Class i but not in Class $i-1$. For instance, the answer to $L = \{a^n b^n \mid n \geq 0\}$ is 3. No explanations are needed.

   (1) $\{a^n b^m a^n b^m \mid m, n \geq 0\}$
   Answer: 4

   (2) $\{w \in \{a, b\}^* \mid$ the length of w is even and the first half is all a's$\}$
   Answer: 3. The language is generated by CFG $S \rightarrow aSa|aSb|\epsilon$.

   (3) $\{w \in \{a, b\}^* \mid$ the number of b's in w is a multiple of the number of a's in w $\}$
   Answer: 4. $L \cap a * b* = \{a^n b^{kn} | n, k \geq 0\}$, which is not context-free.

   (4) $\{w \in \{0, 1\}^* \mid$ the number of times 01 appears as a substring is equal to the number of times 10 appears as a substring $\}$
   Answer: 2.
   The language is $\epsilon + 0(0 + 11^*0)^* + 1(1 + 00^*1)^*$

   (5) $\{\langle M \rangle \mid$ there are at least two strings that M accepts$\}$
   Answer: 5

(6) The complement of $\{a^n b^n c^n \mid n \geq 0\}$
Answer: 3

(7) $\{x_1\#x_2\#\cdots\#x_k \mid k \geq 2, x_h \in \{a,b\}^*, 1 \leq h \leq k$, and $x_i x_k = x_j^R$ for some $i < j < k\}$ (here R denotes "reversal")
Answer: 4. The language intersects with $a^*\#b^*a^*\#b^*$ is $\{a^n\#b^m a^n\#b^m \mid m,n \geq 0\}$, which is not CF.

(8) $\{\langle M_1, M_2\rangle \mid L(M_1) \subseteq L(M_2)$, where M_1, M_2 are Turing machines$\}$
Answer: 6

(9) $\{\langle M_i, M_j, x\rangle \mid M_i(x) \prec M_j(x)\}$, where $M_i(x) \prec M_j(x)$ denotes that $M_i(x)$ halts in fewer steps than $M_j(x)$. We do not specify whether $M_i(x)$ accepts or rejects, and we allow the possibility that $M_j(x)$ never halts.
Answer: 5. Given input $\langle M_i, M_j, x\rangle$ we can first simulate $M_i(x)$ until it halts. If it never halts, $M_i(x) \prec M_j(x)$ is necessarily false, so it's okay if we loop in this phase. If $M_i(x)$ halts after n steps, we then simulate $M_j(x)$ for up to $n+1$ steps. If $M_j(x)$ is still running after $n+1$ steps, we accept.

(10) $\{\langle M_i, M_j\rangle \mid \exists x, M_i(x) \prec M_j(x)\}$, where $\prec$ is defined as above.
Answer: 5. On input $\langle M_i, M_j\rangle$ we enumerate all pairs $\langle x, n\rangle$. For each pair, if $M_i(x)$ halts within n steps and $M_j(x)$ does not, we accept.

3. (5 pts) For language $D = \{w\#w \mid w \in \Sigma^*\}$, it is known that $\overline{D}$ (i.e., the complement of D) is context-free. Use closure properties of context free languages to prove that for any regular language A, the complement of $E = \{w\#w \mid w \in A\}$ (i.e., $\overline{E}$) is context-free.
Answer: $E = D \cap (A \cdot \{\#\} \cdot A)$. Hence, $\overline{E} = \overline{D} \cup \overline{(A \cdot \{\#\} \cdot A)}$, which is the union of a CFL and a regular set – a CFL.

4. (8 pts) Assuming that A is recursive, is the following language B also recursive? Justify your answer formally.
$B = \{w \in \{0,1\}^* \mid \exists x \in \{0,1\}^*, |x| = |w|$ and $x \in A\}$.
Answer: Suppose TM M decides A. Define TM N = On input a string w

Let $n = |w|$ be the length of w
  For all n-bit binary strings x
    do If $M(x)$ accepts then accept
  End-for
Reject

5. (8 pts) Suppose that you are given an algorithm A_F to decide the following language $F = \{\langle Q\rangle \mid$ TM Q halts on at least one input$\}$. Using A_F as a subroutine, give an algorithm A_H to decide $H = \{\langle P, w\rangle \mid$ TM P halts on input $w\}$.
Answer: Given $\langle P, w\rangle$, we construct the following TM Q: on input x
if $x \neq w$ loop forever else Simulate P on x. Then run A_F on input $\langle Q\rangle$.

6. (15 pts) Consider $REGULAR_{TM} = \{\langle M\rangle \mid$ the language recognized by Turing machine M is regular$\}$. Prove that $REGULAR_{TM}$ is neither co-r.e. nor r.e.
Answer See Figure 1.

7. (10 pts) A context-free grammar is ambiguous if some string has two different derivation trees using this grammar. Show that ambiguity of CFGs is undecidable using a reduction from the Post Correspondence Problem.
Answer: Given a PCP instance $P = \{\frac{u_1}{v_1}, \frac{u_2}{v_2}, ..., \frac{u_n}{v_n}\}$, construct the following CFG G:
$S \to S_u \mid S_v;\quad S_u \to u_i S_u a_i \mid u_i a_i;\quad S_v \to v_i S_v a_i \mid v_i a_i, 1 \leq i \leq n.$

8. (8 pts) Prove that recursive languages are NOT closed under homomorphism. (Hint: Given a TM M and an input w, it is undecidable to decide whether M accepts w; however, if a number n is given, then it becomes decidable to decide whether M accepts w in n steps. You may assume that $\langle M, w\rangle$ are encoded using symbols a and b (i.e., $\langle M, w\rangle \in \{a,b\}^*$), and $n \in \{0,1\}^*$ is encoded in binary.)
Answer: Consider $L = \{\langle M, x, n\rangle \mid \langle M, x\rangle \in \{a,b\}^*, n \in \{0,1\}^*,$ TM M on input w will halt in n (represented in a binary number) steps$\}$. Clearly L is recursive. Now consider homomorphism $h(a) = a; h(b) = b; h(0) = h(1) = \epsilon$. Then $h(L) = \{\langle M, x\rangle \mid \langle M, x\rangle \in \{a,b\}^*,$ TM M on input w will halt.$\}$, which is A_{TM}.

We show that $A_{TM} \leq_m REGULAR_{TM}$, and that $A_{TM} \leq_m \overline{REGULAR_{TM}}$, which will prove that it is neither.

$A_{TM} \leq_m REGULAR_{TM}$:

"On input $\langle M, w \rangle$:

Form a TM R as follows:

"On input x:

If x is of the form $0^n 1^n$, accept.

Otherwise run M on w and accept if M accepts w."

Output $\langle R \rangle$."

Then R accepts a regular language if and only if M accepts w.

$A_{TM} \leq_m \overline{REGULAR_{TM}}$:

"On input $\langle M, w \rangle$:

Form a TM S as follows:

"On input x:

If x is not of the form $0^n 1^n$, reject.

Otherwise run M on w and accept if M accepts w."

Output $\langle S \rangle$."

Then S accepts a regular language if and only if M does not accept w.

Figure 1: Proof of Problem 6