Theory of Computation

Final Exam, Fall 2009

Name: .... Student I.D. ....*KEY*...................

1. (24 pts) Determine whether the following classes of languages are closed under the specified operations. (Mark '*O*' for yes; '×' for no. $Score = \{Right - \frac{1}{2}Wrong\}$.)

	CFL	recursive	r.e.
$\cap$	×	O	O
$\cup$	O	O	O
* (star)	O	O	O
complementation	×	O	×
reversal	O	O	O
concatenation	O	O	O
difference	×	O	×
homomorphism	O	×	O

(NOTE: To show recursive languages not closed under homomorphism, consider the language L={$\langle M, w, a^i\rangle$ | TM M accepts w in i steps }, which is recursive. However, if a homomorphism h maps a to ϵ while keeping the remaining symbols intact, $h(L)$={$\langle M, w\rangle$ | TM M accepts w} which is not recursive.

2. (27 pts) Determine whether the following problems are decidable for the specified classes of languages. (Mark '*O*' for decidable; '×' for undecidable. $Score = \{Right - \frac{1}{2}Wrong\}$.)

	CFL	recursive	r.e.
$w \in L$?	O	O	×
$L \neq \emptyset$?	O	×	×
$L = \Sigma^*$?	×	×	×
$L_1 = L_2$?	×	×	×
$L_1 \subseteq L_2$?	×	×	×
$L_1 \cup L_2 = \Sigma^*$?	×	×	×
$L_1 \cap L_2 = \emptyset$?	×	×	×
Is $L_1 = R$, for some given regular language R?	×	×	×
Is $L_1 \cap R = \emptyset$, for some given regular language R?	O	×	×

3. (15 pts)
(a) (5 pts) Define *(many-one) mapping reduction* $L \leq_m L'$ formally, where $L \subseteq \Sigma^*$ and $L' \subseteq \Gamma^*$.
(Ans.) **There exists a total computable function $f : \Sigma^* \to \Gamma^*$ such that $w \in L$ iff $f(w) \in L'$.**

(b) (10 pts) Let $B = \{\langle M\rangle \mid M$ is a Turing machine and $L(M) \neq \{0^n1^n \mid n \geq 0\}\}$. Suppose we want to show that $L_u \leq_m B$, where $L_u = \{\langle M, w\rangle \mid$ Turing machine M accepts $w\}$. We use the following mapping from (M, w) to M':

Given M and w, construct M' = "On input x"
(1) If $x \in \{0^n1^n \mid n \geq 0\}$, M' .....*ACCEPTS*.......................
(2) If $x \notin \{0^n1^n \mid n \geq 0\}$, run M on w
If M .....*ACCEPTS*......................., then M' .....*ACCEPTS*.......................

Fill in "accepts" or 'rejects" for each of the above three blanks.
Then we can conclude that

- If M accepts w, then $L(M')$ .... $\neq \{0^n1^n \mid n \geq 0\}$.....................................
- If M does not accept w, then $L(M')$ .... $= \{0^n1^n \mid n \geq 0\}$.....................................

4. (9 pts) Given an alphabet Σ, is it true that $L(\subseteq \Sigma^*)$ is recursive if and only if $L \leq_m 011^*$? Justify your answer in a rigorous way.
(Ans.) **Define the following function $f : \Sigma^* \to \{01, 11\}$ with $f(w) = 01$ if $w \in L$; otherwise, $f(w) = 11$. Clearly f is a total computable function since L is recursive. (One can think of f as a TM that decides the membership of L, and if accepts (resp., rejects), outputs 01 (resp., 11).) Hence, $L \leq_m 011^*$.**

5. (10 pts) A *palindrome* is a string x such that $x = x^R$ (e.g., *abccba*). Given a context-free language L, show that it is undecidable whether L contains a palindrome. (Hint: use PCP.)
(Ans.) **Let $\{(x_i, y_i) \mid 1 \leq i \leq n\}$ be an instance of the PCP problem, where $x_i, y_i \in \Sigma^*$. We construct the following CFG $G = (V, T, P, S)$**

   (a) $V = \{S, S_xS_y\} \cup \{X_i, Y_i, 1 \leq i \leq n\}$

   (b) $T = \{a_i, 1 \leq i \leq n\} \cup \{\#\} \cup \Sigma$

   (c) P:

   - $S \to S_x\#S_y$
   - $S_x \to a_iS_xx_i \mid a_ix_i$, $1 \leq i \leq n$
   - $S_y \to (y_i)^RS_ya_i \mid (y_i)^Ra_i$, $1 \leq i \leq n$

   It is not hard to see that G generates a palindrome iff the PCP instance has a solution.

6. (15 pts)
(a) (5 pts) State *Ogden's lemma.*
(b) (10 pts) Use Ogden's lemma to prove that $L = \{a^nb^nc^i \mid i \neq n\}$ is not context-free. (Hint: consider a string of the form $a^nb^nc^{n+n!}$.)
(Ans.) **If $L \subseteq \Sigma^*$ is context-free, then there exists a constant n such that for every $w \in L$ with $|w| \geq n$ and we mark n or more positions of w, w can be written as $w = uvxyz$, for some u, v, x, y, z, such that**

   (a) **v and y together have at least one marked position,**

   (b) **vxy has at most n marked positions, and**

   (c) **for all $i \geq 0$, $uv^ixy^iz \in L$**

   Assume that $L = \{a^nb^nc^i \mid i \neq n\}$ were context-free. Let n be the constant guaranteed by Ogden's lemma. Consider $a^nb^nc^{n+n!}$ ($\in L$), and we mark all the a's in a^n. Suppose $a^nb^nc^{n+n!} = uvxyz$. Consider the following cases:

   (a) **v (or y) contains more than one type of symbols. Then uv^2xy^2z is clearly not in L,**

   (b) **vxy is in a^n. Then uv^2xy^2z contains more a than b, which is clearly not in L,**

   (c) **v is in a^n and y is in b^n but $|v| \neq |y|$. Clearly uv^2xy^2z is not in L,**

   (d) **v is in a^n and y is in b^n and $|v| = |y| = p$, where $1 \leq p < n$. Let $v = a^p$, $y = b^p$. Consider $uv^{1+\frac{n!}{p}}xy^{1+\frac{n!}{p}}z = a^{n-p}(a^p)^{1+\frac{n!}{p}}b^{n-p}(b^p)^{1+\frac{n!}{p}}c^{n+n!} = a^{n+n!}b^{n+n!}c^{n+n!} \notin L$.**