Theory of Computation
Reducibility

Reducibility

- In mathematics, many problems are solved by "reduction."
- Recall the reduction from Eulerian path to Eulerian cycle.
  - Suppose $EC(G)$ returns true iff G has a Eulerian cycle.
  - Let s, t be nodes of a graph G.
  - To check if there is a Eulerian path from s to t in G.
  - Construct a graph G' that is identical to G except an additional edge between s and t.
  - If $EC(G')$ returns true, there is a Eulerian path from s to t.
  - If $EC(G')$ returns false, there is no Eulerian path from s to t.
- Instead of inventing a new algorithm for finding Eulerian paths, we use $EC(G)$ as a subroutine.
- We say the Eulerian path problem is reduced to the Eulerian cycle problem.

Reducibility

- Let us say A and B are two problems and A is reduced to B.
- If we solve B, we solve A as well.
  - If we solve the Eulerian cycle problem, we solve the Eulerian path problem.
- If we can't solve A, we can't solve B.
- To show a problem P is not decidable, it suffices to reduce A_{TM} to P.
- We will give examples in this chapter.

The Halting Problem for Turing Machines

- The halting problem is to test whether a TM M halts on a string w.
- As usual, we first give a language-theoretic formulation.

$$HALT_{\text{TM}} = \{\langle M, w \rangle : M \text{ is a TM and } M \text{ halts on the input } w\}.$$

Theorem 1

$HALT_{TM}$ is undecidable.

Proof.

We would like to reduce the acceptance problem to the halting problem. Suppose a TM R decides $HALT_{\text{TM}}$. Consider
$S =$ "On input $\langle M, w \rangle$ where M is a TM and w is a string:

1. Run TM R on the input $\langle M, w \rangle$.
2. If R rejects, reject.
3. If R accepts, simulate M on w until it halts.
4. If M accepts, accept; if M rejects, reject." □

Emptiness Problem for Turing Machines

- Consider $E_{\mathrm{TM}} = \{\langle M \rangle : M \text{ is a TM and } L(M) = \emptyset\}$.

Theorem 2

E_{TM} is undecidable.

Proof.

We reduce the acceptance problem to the emptiness problem. Let the TM R decides E_{TM}. Consider
$S =$ "On input $\langle M, w \rangle$ where M is a TM and w a string:

1. Use $\langle M \rangle$ to construct
   $M_1 =$ "On input x:
   1. If $x \neq w$, reject.
   2. If $x = w$, run M on the input x. If M accepts x, accept."
2. Run R on the input $\langle M_1 \rangle$.
3. If R accepts, reject; otherwise, accept." □

Regularity Problem for Turing Machines

- Consider

$$REGULAR_{\mathrm{TM}} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular}\}.$$

Theorem 3

$REGULAR_{TM}$ is undecidable.

Proof.

Let R be a TM deciding $REGULAR_{\mathrm{TM}}$. Consider
S = "On input $\langle M, w \rangle$ where M is a TM and w a string:

1. Use $\langle M \rangle$ to construct
   M_2 = "On input x:
   1. If x is of the form $\mathtt{0}^n\mathtt{1}^n$, accept.
   2. Otherwise, run M on the input w. If M accepts w, accepts."
2. Run R on the input $\langle M_2 \rangle$.
3. If R accepts, accept; otherwise, reject." □

Rice's Theorem

Theorem 4

Let P be a language consisting of TM descriptions such that

1. *P is not trivial ($P \neq \emptyset$ and there is a TM M with $\langle M \rangle \notin P$);*
2. *If $L(M_1) = L(M_2)$, $\langle M_1 \rangle \in P$ iff $\langle M_2 \rangle \in P$.*

Then P is undecidable.

Proof.

Let R be a TM deciding P. Let $T_\emptyset$ be a TM with $L(T_\emptyset) = \emptyset$. WLOG, assume $\langle T_\emptyset \rangle \notin P$. Moreover, pick a TM T with $\langle T \rangle \in P$. Consider
$S =$ "On input $\langle M, w \rangle$ where M is a TM and w a string:

1. Use $\langle M \rangle$ to construct

   $M_w =$ "On input x:

   1. Run M on w. If M halts and rejects, reject.
   2. If M accepts w, run T on x."

2. Run R on $\langle M_w \rangle$.
3. If R accepts, accept; otherwise, reject."

□

Language Equivalence Problem for Turing Machines

- Consider

$$EQ_{\mathrm{TM}} = \{\langle M_1, M_2 \rangle : M_1 \text{ and } M_2 \text{ are TM's with } L(M_1) = L(M_2)\}.$$

Theorem 5

EQ_{TM} is undecidable.

Proof.

We reduce the emptiness problem to the language equivalence problem this time. Let the TM R decide EQ_{TM} and TM M_1 with $L(M_1) = \emptyset$. Consider

$S =$ "On input $\langle M \rangle$ where M is a TM:

1. Run R on $\langle M, M_1 \rangle$.
2. If R accepts, accept; otherwise, reject." □

Computation History

Definition 6

Let M be a TM and w an input string. An accepting computation history for M on w is a sequence of configurations $C_1, C_2, \ldots, C_l$ where

- C_1 is the start configuration of M on w;
- C_l is an accepting configuration of M; and
- C_i yields C_{i+1} in M for $1 \leq i < l$.

A rejecting computation history for M on w is similar, except C_l is a rejecting configuration.

- Note that a computation history is a finite sequence.
- A deterministic Turing machine has at most one computation history on any given input.
- A nondeterminsitic Turing machine may have several computation histories on an input.

Languages Associated with Computation Histories

Suppose $\alpha \vdash \beta$ is a single step of a TM M. We consider the following cases (examples):

	left move	right move
α	$abcdqefgh$	$abcdqefgh$
β	$abcq'de'fgh$	$abcde'q'fgh$

Notice that in α and β, at most 3 positions may change.
Consider accepting computation $\alpha_0 \vdash \alpha_1 \vdash \alpha_2 \vdash \alpha_3 \vdash \cdots \vdash \alpha_n$

- CS: $\alpha_0\#\alpha_1\#\alpha_2\#\alpha_3\#\cdots\#\alpha_n$
- CS_R: $\alpha_0\#\alpha_1^R\#\alpha_2\#\alpha_3^R\#\cdots\#\alpha_n$

CS_R is the intersection of two CFL L_{odd} and L_{even}, where

- $L_{odd} = \{\alpha_0\#\alpha_1^R\#\alpha_2\#\alpha_3^R\#\cdots\#\alpha_n \mid \alpha_i \vdash \alpha_{i+1}, i \text{ is odd}\}$
- $L_{even} = \{\alpha_0\#\alpha_1^R\#\alpha_2\#\alpha_3^R\#\cdots\#\alpha_n \mid \alpha_i \vdash \alpha_{i+1}, i \text{ is even}\}$

Linear Bounded Automaton

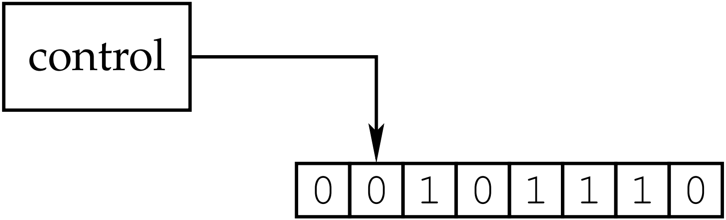


Figure: Schematic of Linear Bounded Automata

Definition 7

A linear bounded automaton is a Turing machine whose tape head is not allowed to move off the portion of its input. If an LBA tries to move its head off the input, the head stays.

- With a larger tape alphabet than its input alphabet, an LBA is able to increase its memory up to a constant factor.

Acceptance Problem for Linear Bounded Automata

- Consider

$$A_{\text{LBA}} = \{\langle M, w\rangle : M \text{ is an LBA and } M \text{ accepts } w\}.$$

Lemma 8

Let M be an LBA with q states and g tape symbols. There are exactly qng^n different configurations of M for a tape of length n.

- An LBA has only a finite number of different configurations on an input.
- Many langauges can be decided by LBA's.
  - For instance, $A_{\text{DFA}}, A_{\text{CFG}}, E_{\text{DFA}}$, and E_{CFG}.
- Every context-free langauges can be decided by LBA's.

Acceptance Problem for Linear Bounded Automata

Theorem 9

A_{LBA} is decidable.

Proof.

Consider

$L =$ "On input $\langle M, w\rangle$ where M is an LBA and w a string:

1. Simulate M on w for qng^n steps or until it halts. (q, n, and g are obtained from $\langle M\rangle$ and w.)
2. If M does not halt in qng^n steps, reject.
3. If M accepts w, accept; if M rejects w, reject." □

- The acceptance problem for LBA's is decidable. What about the emptiness problem for LBA's?

$$E_{\text{LBA}} = \{\langle M\rangle : M \text{ is an LBA with } L(M) = \emptyset\}.$$

Emptiness Problem for Linear Bounded Automata

Theorem 10

E_{LBA} is undecidable.

Proof.

We reduce the acceptance problem for TM's to the emptiness problem for LBA. Let R be a TM deciding E_{LBA}. Consider
$S =$ "On input $\langle M, w \rangle$ where M is a TM and w a string:

1. Use $\langle M \rangle$ to construct the following LBA:
   $B =$ "On input $\langle C_1, C_2, \ldots, C_l \rangle$ where C_i's are configurations of M:
   1. If C_1 is not the start configuration of M on w, reject.
   2. If C_l is not an accepting configuration, reject.
   3. For each $1 \leq i < l$, if C_i does not yield C_{i+1}, reject.
   4. Otherwise, accept."
2. Run R on $\langle B \rangle$.
3. If R rejects, accept; otherwise, reject." □

Context Sensitive Grammars

- A context sensitive grammar (CSG) is a grammar where all productions are of the form

$$\alpha A\beta \to \alpha\gamma\beta,\ \ \alpha,\beta \in (N \cup \Sigma)^*, \gamma \in (N \cup \Sigma)^+,$$

- During derivation non-terminal A will be replaced by γ only when it is present in context of α and β.
- This definition shows clearly one aspect of this type of grammar; it is noncontracting, in the sense that the length of successive sentential forms can never decrease.
- The production $S \to \epsilon$ is also allowed if S is the start symbol and it does not appear on the right side of any production.
- A language L is said to be context-sensitive if there exists a context-sensitive grammar G, such that $L = L(G)$.
- An alternative definition of CSG:

$$u \to v,\ \ |u| \le |v|, u, v \in (N \cup \Sigma)^+,$$

An Example

$\{a^n b^n c^n \mid n \geq 1\}$ is a CSL.

$S \to \Lambda \mid abc \mid aTBc$

$T \to abC \mid aTBC$

$CB \to CX \to BX \to BC$

$bB \to bb.$

$Cc \to cc.$

Ex: $S \Rightarrow aTBc \Rightarrow aaTBCBc \Rightarrow aaabCBCBc \Rightarrow aaabBCCBc$
$\Rightarrow aaabBCBCc \Rightarrow aaabBBCCc \Rightarrow aaabbBCCc \Rightarrow aaabbbCCc$
$\Rightarrow aaabbbCcc \Rightarrow aaabbbccc.$

More on CSLs

CSLs are closed under

- Union
- Intersection
- Complement
  Immerman-Szelepcsenyi theorem (1987).
- Concatenation
- Kleene closure

Theorem 11

A language is context-sensitive iff it can be accepted by a linear-bounded automaton.

Universality of Context-Free Grammars

- Consider a problem related to the emptiness problem for CFL's

$$ALL_{\text{CFG}} = \{\langle G\rangle : G \text{ is a CFG and } L(G) = \Sigma^*\}.$$

- Let x be a string. Write x^R for the string x in reverse order.
  - For example, $\texttt{100}^R = \texttt{001}$, $\texttt{level}^R = \texttt{level}$.
  - Another example,

乾隆：　客上天然居 居然天上客
紀曉嵐：　人過大鐘寺 寺鐘大過人

- Let $C_1, C_2, \ldots, C_l$ be the accepting configuration of M on input w. Consider the following string in the next theorem:

$$\#\langle C_1\rangle\#\langle C_2\rangle^R\#\cdots\#\langle C_{2k-1}\rangle\#\langle C_{2k}\rangle^R\#\cdots\#\langle C_l\rangle\#$$

Universality of Context-Free Grammars

Theorem 12

ALL_{CFG} is undecidable.

Proof.

We reduce the acceptance problem for TM's to the universalty problem. We construct a nondeterministic PDA D that accepts all strings if and only if M does not accept w. The input and stack alphabets of D contain symbols to encode M's configurations.

$D =$ "On input $\#x_1\#x_2\# \cdots \#x_l\#$:

1. Do one of the following branches nondeterministically:
   - If $x_1 \neq \langle C_1 \rangle$ where C_1 is the start configuration of M on w, accept.
   - If $x_l \neq \langle C_l \rangle$ where C_l is a rejecting configuration of M, accept.
   - Choose odd i nondeterministically. If $x_i \neq \langle C \rangle$, $x_{i+1}^R \neq \langle C' \rangle$, or C does not yield C' (C, C' are configurations of M), then accept."
   - Choose even i nondeterministically. If $x_i^R \neq \langle C \rangle$, $x_{i+1} \neq \langle C' \rangle$, or C does not yield C' (C, C' are configurations of M), then accept."

M accepts w iff the accepting computation history of M on w is not in $L(D)$ iff $CFG(D) \notin ALL_{CFG}$. □

Post Correspondence Problem (PCP)

- A <u>domino</u> is a pair of strings: $\left[\frac{t}{b}\right]$
- A <u>match</u> is a sequence of dominos $\left[\frac{t_1}{b_1}\right]\left[\frac{t_2}{b_2}\right]\cdots\left[\frac{t_k}{b_k}\right]$ such that $t_1 t_2 \cdots t_k = b_1 b_2 \cdots b_k$.
- The <u>Post correspondence problem</u> is to test whether there is a match for a given set of dominos.

$$PCP = \{\langle P \rangle : P \text{ is an instance of the PCP with a match}\}$$

- Consider

$$P = \left\{ \left[\frac{\texttt{b}}{\texttt{ca}}\right], \left[\frac{\texttt{a}}{\texttt{ab}}\right], \left[\frac{\texttt{ca}}{\texttt{a}}\right], \left[\frac{\texttt{abc}}{\texttt{c}}\right] \right\}$$

- A match in P:

$$\left[\frac{\texttt{a}}{\texttt{ab}}\right]\left[\frac{\texttt{b}}{\texttt{ca}}\right]\left[\frac{\texttt{ca}}{\texttt{a}}\right]\left[\frac{\texttt{a}}{\texttt{ab}}\right]\left[\frac{\texttt{abc}}{\texttt{c}}\right]$$

The Modified Post Correspondence Problem

- The modified Post correspondence problem is a PCP where a match starts with the first domino. That is,

$$MPCP = \{\langle P \rangle : \ P \text{ is an instance of the PCP with a match starting with the first domino}\}$$

Theorem 13

PCP is undecidable.

Proof idea.

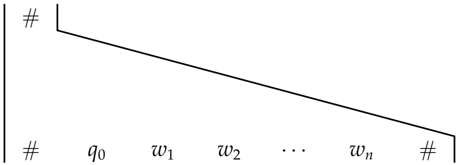
We reduce the acceptance problem for TM's to PCP. Given a TM M and a string w, we first construct an MPCP P' such that $\langle P' \rangle \in MPCP$ if and only if M accepts w. The MPCP P' encodes an accepting computation history of M on w. Finally, we reduce MPCP P' to PCP P.

The Post Correspondence Problem

Proof.

Let the TM R decide $MPCP$. Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$ be the given TM and $w = w_1 w_2 \cdots w_n$ the input. The set P' of dominos has

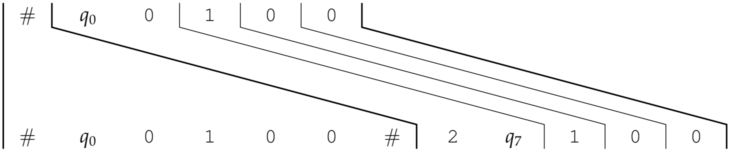
- $\left[\dfrac{\#}{\# q_0 w_1 w_2 \cdots w_n \#}\right]$ as the first domino. Begin with the start configuration (bottom).



The Post Correspondence Problem

Proof (cont'd).

- $\left[\dfrac{qa}{br}\right]$ if $\delta(q,a) = (r,b,R)$ with $q \neq q_{\text{reject}}$. Reads a at state q (top); writes b and moves right (bottom).
- $\left[\dfrac{cqa}{rcb}\right]$ if $\delta(q,a) = (r,b,L)$ with $q \neq q_{\text{reject}}$. Reads a at state q (top); writes b and moves left (bottom).
- $\left[\dfrac{a}{a}\right]$ if $a \in \Gamma$. Keeps other symbols intact.

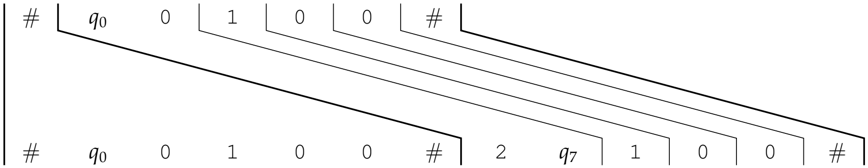


$$\delta(q_0, 0) = (q_7, 2, R)$$

The Post Correspondence Problem

Proof (cont'd).

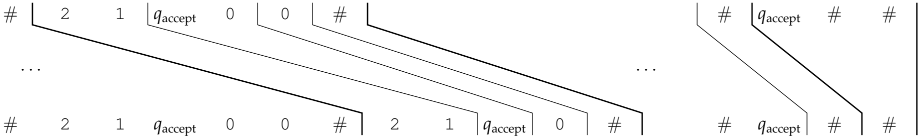
- $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ Matches previous # (top) with a new # (bottom). Adds $\sqcup$ when M moves out of the right end.



The Post Correspondence Problem

Proof (cont'd).

- $\left[\frac{aq_{\text{accept}}}{q_{\text{accept}}}\right]$ and $\left[\frac{q_{\text{accept}}a}{q_{\text{accept}}}\right]$ if $a \in \Gamma$. Eats up tape symbols around q_{accept}.
- $\left[\frac{q_{\text{accept}}\#\#}{\#}\right]$. Completes the match.



Proof (cont'd).

So far, we have reduced the acceptance problem of TM's to MPCP. To complete the proof, we need to reduce MPCP to PCP.
Let $u = u_1 u_2 \cdots u_n$. Define

$$\begin{array}{rcccccccccc} \star u & = & * & u_1 & * & u_2 & * & \cdots & * & u_n & \\ u\star & = & & u_1 & * & u_2 & * & \cdots & * & u_n & * \\ \star u \star & = & * & u_1 & * & u_2 & * & \cdots & * & u_n & * \end{array}$$

Given a MPCP P':

$$\left\{ \left[\frac{t_1}{b_1} \right], \left[\frac{t_2}{b_2} \right], \ldots, \left[\frac{t_k}{b_k} \right] \right\}$$

Construct a PCP P:

$$\left\{ \left[\frac{\star t_1}{\star b_1 \star} \right], \left[\frac{\star t_2}{b_2 \star} \right], \ldots, \left[\frac{\star t_k}{b_k \star} \right], \left[\frac{* \Diamond}{\Diamond} \right] \right\}$$

Any match in P must start with the domino $\left[\frac{\star t_1}{\star b_1 \star} \right]$. □

Computable Functions

Definition 14

$f : \Sigma^* \to \Sigma^*$ is computable if some Turing machine M, on input w, halts with $f(w)$ on its tape.

- Usual arithmetic operations on integers are computable functions. For instance, the addition operation is a computable function mapping $\langle m, n \rangle$ to $\langle m + n \rangle$ where m, n are integers.

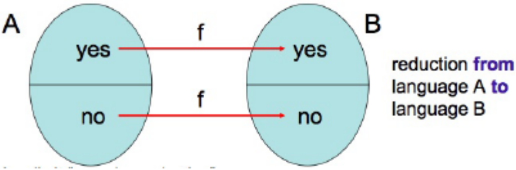
Mapping Reducibility

Definition 15

A language A is mapping reducible (or many-one reducible) to a languate B (written $A \leq_m B$) if there is a computable function $f : \Sigma^* \rightarrow \Sigma^*$ such that

$$w \in A \text{ if and only if } f(w) \in B, \text{ for every } w \in \Sigma^*.$$

f is called the reduction of A to B.



Properties of Reducibility

Theorem 16

If $A \leq_m B$ and B is decidable, A is decidable.

Proof.

Let the TM M decide B and f the reduction of A to B. Consider
$N =$ "On input w:

1. Construct $f(w)$.
2. Run M on $f(w)$.
3. If M accepts, accept; otherwise reject.

□

Corollary 17

If $A \leq_m B$ and A is undecidable, then B is undecidable.

Example 18

Give a mapping reduction of A_{TM} to $HALT_{\text{TM}}$.

Proof.

We need to show a computable function f such that $\langle M, w\rangle \in A_{\text{TM}}$ if and only if $\langle M', w'\rangle \in HALT_{\text{TM}}$ whenever $\langle M', w'\rangle = f(\langle M, w\rangle)$.
Consider
$F =$ "On input $\langle M, w\rangle$:

1. Use $\langle M\rangle$ and w to construct
   $M' =$ "On input x:
   1. Run M on x.
   2. If M accepts, accept.
   3. If M rejects, loop."
2. Output $\langle M', w\rangle$."

□

Examples

Example 19

Give a mapping reduction of A_{TM} to $Regular_{TM} = \{\langle M \rangle \mid L(M) \text{ is regular}\}$.

- $f(\langle M, w \rangle) = \langle M' \rangle$ described below

M' takes input x:

- if x has form 0^n1^n, accept
- else simulate M on w and accept x if M accepts

$M' = \{0^n1^n\}$ if $w \notin L(M)$
$= \Sigma^*$ if $w \in L(M)$

What would a formal proof of this look like?

- is f computable?
- YES maps to YES?
  $\langle M, w \rangle \in ACC_{TM} \Rightarrow f(M, w) \in REGULAR$
- NO maps to NO?
  $\langle M, w \rangle \notin ACC_{TM} \Rightarrow f(M, w) \notin REGULAR$

Examples

Example 20

Give a mapping reduction from E_{TM} to EQ_{TM}.

Proof.

The proof of Theorem 5 gives such a reduction. The reduction maps the input $\langle M \rangle$ to $\langle M, M_1 \rangle$ where M_1 is a TM with $L(M_1) = \emptyset$. $\square$

Transitivity of Mapping Reductions

Lemma 21

If $A \leq_m B$ and $B \leq_m C$, $A \leq_m C$.

Proof.

Let f and g be the reductions of A to B and B to C respectively. $g \circ f$ is a reduction of A to C. □

Example 22

Give a mapping reduction from A_{TM} to PCP.

Proof.

The proof of Theorem 13 gives such a reduction. We first show $A_{\text{TM}} \leq_m MPCP$. Then we show $MPCP \leq_m PCP$. □

More Properties about Mapping Reductions

Theorem 23

If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

Proof.

Similar to the proof of Theorem 16 except that M and N are TM's, not deciders. □

Corollary 24

If $A \leq_m B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

More Properties about Mapping Reductions

- Observe that $A \leq_m B$ if and only if $\overline{A} \leq_m \overline{B}$.
  - The same reduction applies to $\overline{A}$ and $\overline{B}$ as well.
- Recall that $\overline{A_{\text{TM}}}$ is not Turing-recognizable.
- In order to show B is not Turing-recognizable, it suffices to show $A_{\text{TM}} \leq_m \overline{B}$.
  - $A_{\text{TM}} \leq_m \overline{B}$ implies $\overline{A_{\text{TM}}} \leq_m \overline{\overline{B}}$. That is, $\overline{A_{\text{TM}}} \leq_m B$.

Equivalence Problem for TM's (revisited)

Theorem 25

EQ_{TM} is neither Turing-recognizable nor co-Turing-Recognizable.

Proof.

We first show $A_{\text{TM}} \leq_m \overline{EQ_{\text{TM}}}$. Consider
$F =$ "On input $\langle M, w \rangle$ where M is a TM and w a string:

1. Construct
   $M_1 =$ "On input x:
   1. Reject."

   $M_2 =$ "On input x:
   1. Run M on w. If M accepts, accept."
2. Output $\langle M_1, M_2 \rangle$."

Equivalence Problem for TM's (revisited)

Proof (cont'd).

Next we show $A_{\text{TM}} \leq_m EQ_{\text{TM}}$. Consider
$G =$ "On input $\langle M, w \rangle$ where M is a TM and w a string:

1. Construct
   $M_1 =$ "On input x:
   1. Accept."

   $M_2 =$ "On input x:
   1. Run M on w.
   2. If M accepts w, accept."
2. Output $\langle M_1, M_2 \rangle$."

□