# Theory of Computation

Spring 2023, Homework #2 Solution

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1.

Suppose $L = \{a^n b^j c^k \mid k = jn\}$ is context free and $m$ is the pumping length. Let $s = a^m b^m c^{m^2} \in L$. $|s| = m^2 + 2m > m$. According to pumping lemma, there exists a partition $s = uvxyz$ where (1) $|vy| > 0$; (2) $|vxy| \leq m$; and (3) $uv^i xy^i z \in L$ for $i \geq 0$. If either $v$ or $y$ contains more than one type of symbols, then clearly $uv^2xy^2z \notin L$. So both $v$ and $y$ can only contain one type of symbols. Consider the following two cases:

(1) $v$ and $y$ contain the same type of symbols. Since $|vy| > 0$, $uv^0xy^0z \notin L$.

(2) $v$ and $y$ contain different types of symbols. Since $|vxy| \leq m$, $(v, y) = (a*, b*)$ or $(b*, c*)$. In both cases, $uv^0xy^0z \notin L$. (E.g., $v = a*; y = b*$, $s' = uv^0xy^0z = a^{m-|v|}b^{m-|y|}c^{m^2}$. Since $(m - |v|) \cdot (m - |y|) \neq m^2$ when $|v| + |y| > 0$, so $s' \notin L$.)

So there is no partition that can satisfy the pumping lemma. Contradiction. So $L$ is not context-free.

2.

Suppose $ADD$ is regular and $p$ is the pumping length. Choose $s =$ "$1^p = 0 + 1^p$". So $|s| = 2p + 3 \geq p$. According to pumping lemma, there exists a partition $s = abc$ where (1) $ab^ic \in ADD$ for $i \geq 0$, (2) $|b| \geq 1$, and (3) $|ab| \leq p$. Given (2) and (3), we have $b = 1^k$ where $k \geq 1$. Therefore $ab^0c =$ "$1^{p-k} = 0 + 1^p$" $\notin ADD$. A contradiction. So $ADD$ is not regular.

3.

$B = L_1 \cup L_2 \cup L_3$ where

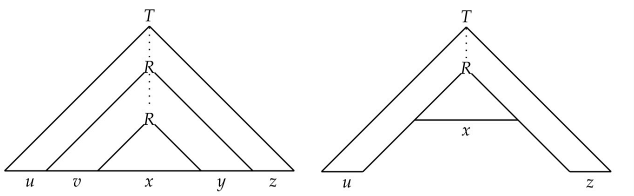
(1) $L_1 = \{a^n b^m \mid n > m\}$. $L_1$ is context-free since it can be generated by the context-free grammar $G_1 = (\{S, A, B\}, \{a, b\}, R, S)$ where $R$ is: $S \to AB,\ A \to aA \mid a$, $B \to aBb \mid \epsilon$.

(2) $L_2 = \{a^n b^m \mid n < m\}$. $L_2$ is context-free since it can be generated by the context-free grammar $G_2 = (\{S, A, B\}, \{a, b\}, R, S)$ where $R$ is: $S \to AB,\ A \to aAb \mid \epsilon$, $B \to Bb \mid b$.

(3) $L_3 = \overline{a*b*}$. $L_3$ is regular and therefore also context-free.

4 (a).

Let $L = (V, \Sigma, R, S)$ be a linear language. Define $b$ to be the maximum number of terminals in the right-hand side of a rule. Based on the definition of linear language, the right-hand side of any rule can contain at most one nonterminal. As a result, all the nonterminals must appear on the same branch in the parse tree. So a parse tree with height $h$ has at most $bh$ leaves (terminals), which is also the maximal length of any strings it can generate.
Choose $p = b(|V| + 1)$. Let $w \in L$ with $|w| \geq p$ and $\tau$ be the smallest parse tree for $w$. Then the height of $\tau \geq |V| + 1$, which means the longest branch in $\tau$ contains $|V| + 1$ nonterminals. So there exists a nonterminal which appears more than once in the branch. Let $R$ be the first nonterminal that appears twice. Figure (a) below shows how the leaves of $\tau$ are partitioned. The root of the subtree generating $vxy$ corresponds to the 1st occurrence of $R$. The root of the subtree generating $x$ corresponds to the 2nd occurrence of $R$.




(a) Smallest parse tree $\tau$ (b) A parse tree if $|vy| = 0$

i) From Figure (a), we see $uv^ixt^iz \in L$ for $i \geq 0$.
ii) If $|vy| = 0$, Figure (b) is a smaller parse tree than $\tau$. A contradiction. Hence $|vy| > 0$.
iii) There are at most $|V| + 1$ nonterminals from $T$ to the 2nd occurrence of $R$, both ends included. (Otherwise $R$ cannot be the "first" nonterminal to appear twice.) Therefore $|uvyz| \leq b|V| < p$.

4 (b).

Suppose $L = \{a^n b^{2n} a^n \mid n \geq 0\}$ is linear and $p$ the pumping length. Choose $w = a^p b^{2p} a^p \in L$. $|w| = 4p \geq p$. Let $w = uvxyz$ be a partition that satisfies the pumping lemma. That is, $|uvzy| \leq p$, $|vy| \geq 1$, and $uv^ixy^iz \in L$ for all $i \geq 0$. Since $|x| = 4p - |uvyz| \geq 3p$, $x$ must contain the entire substring $b^{2p}$. Consider $s' = uv^0xy^0z = a^{p-|v|}b^{2p}a^{p-|y|}$. Since $|v| + |y| > 0$, $s' \notin L$. Contradiction. Hence $L$ is not linear.

5.

$B \propto D$ if $B \subseteq D$ and $D \backslash B$ contains infinitely many strings.

Since both $B$ and $D$ are regular languages, $D \backslash B$ is also regular. Therefore pumping lemma must hold for $D \backslash B$. Let $p$ be the pumping length for $D \backslash B$. Choose a string $s$ from $D \backslash B$ where $|s| \geq p$. According to pumping lemma, there exists a partition $s = xyz$ where

$xy^iz \in D \backslash B$ for any $i \geq 0$. Let $L_{even} = \{ xy^iz \mid i \geq 0$ and $i$ is even$\}$. Clearly $L_{even} \subseteq D \backslash B \subseteq D$ and $L_{even}$ contains infinitely many strings (since there are infinitely many values for $i$). Similarly, we can define $L_{odd} = \{ xy^iz \mid i \geq 0$ and $i$ is odd$\}$. Then $L_{odd} \subseteq D \backslash B \subseteq D$ and $L_{odd}$ contains infinitely many strings. Note that $L_{even} \cap L_{odd} = \emptyset$.

Let $C = L_{even} \cup B$. We first prove that $B \propto C$: $B \subseteq B \cup L_{even} = C$. Since $L_{even} \subseteq D \backslash B$, $L_{even} \cap B = \emptyset$. Therefore $C \backslash B = L_{even}$, which contains infinitely many strings. Therefore $B \propto C$.

We then prove that $C \propto D$: Since $B \subseteq D$ and $L_{even} \subseteq D$, we have $C = L_{even} \cup B \subseteq D$. Since $L_{odd} \subseteq D \backslash B$, $L_{odd} \cap B = \emptyset$. Given that $L_{even} \cap L_{odd} = \emptyset$, we have $L_{odd} \cap C = \emptyset$. So $L_{odd} \subseteq D \backslash C$. We already know that $L_{odd}$ contains infinitely many strings. Therefore $C \propto D$.