Theory of Computation

Spring 2023, Homework #1 Solution

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1 (a)

Let $L = \{x \in \{0,1\}^* \mid \#_0(x) > 0, \#_1(x) > 0, \text{GCD}(\#_0(x), \#_1(x)) = 1\}$. Claim that L is not regular.

Proof: $\overline{L} = \{x \in \{0,1\}^* \mid \#_0(x) > 0, \#_1(x) > 0, \text{GCD}(\#_0(x), \#_1(x)) > 1\}$ is the complement of L by definition. To simplify the proof, we introduce the language L_A where $L_A = \overline{L} \cap L(0^*1^*) = \{\, 0^a1^b \mid a > 0, b > 0, \text{GCD}(a, b) > 1\}$. Since the class of regular languages is closed under complementation and intersection, if L_A is not regular, then L is not regular. We then use pumping lemma to prove that L_A is not regular.

Suppose L_A is a regular language and p the pumping length. Choose $s = 0^r1^r$ where $r \geq p$ and r is a prime number. According to pumping lemma, there is a partition $s = xyz$ such that $|y| > 0$, $|xy| \leq p$, and for each $i \geq 0$, $xy^iz \in L_A$. Consider the following three cases:

i) y contains both 0s and 1s. Clearly $xy^2z \notin L_A$ since 1s cannot appear before 0s.

ii) y contains only 0s. Let $|y| = k$ and $0 < k \leq p$. Then $xy^0z = xz = 0^{r-k}1^r \notin L_A$ since $\text{GCD}(r-k, r) = 1$.

iii) y contains only 1s. Let $|y| = k$ and $0 < k \leq p$. Then $xy^0z = xz = 0^r1^{r-k} \notin L_A$ since $\text{GCD}(r, r-k) = 1$.

Since one of the above three cases must be true, we can conclude that the pumping lemma cannot be satisfied here. Therefore L_A is not regular.

1 (b)

Let $L = \{w \in \{0,1\}^* \mid (\#_0(w) \mod 3) = (\#_1(w) \mod 3)\}$. Claim that L is regular.

Proof: For each i in $\{0, 1, 2\}$, let $L_i^0 = L(0^i \cdot (000)^*)$. Then $\#_0(w) \mod 3 = i$ for any string $w \in L_i^0$. Similarly, let $L_i^1 = L(1^i \cdot (111)^*)$. Then $\#_1(w) \mod 3 = i$ for any string $w \in L_i^1$. Since they are described by regular expressions, L_i^0 and L_i^1 are both regular.

Let $L_i = L_i^0 \parallel L_i^1$, where $\parallel$ is the shuffle operator. Since the class of regular languages is closed under shuffle, L_i is also regular.

Observe that for any string $w \in L_i$, $(\#_0(w) \mod 3) = (\#_1(w) \mod 3) = i$. That is, $L_i = \{w \in \{0,1\}^* \mid (\#_0(w) \mod 3) = (\#_1(w) \mod 3) = i\,\}$. Clearly $L = L_0 \cup L_1 \cup L_2$. Since the class of regular languages is closed under the operation of union, L is regular.

2.

Let $A = \{0^{i+1}1\,0^i\,1 \mid i \geq 1\,\}$. The set $\{0^n\,1\,0^{n-1}\,1\,0^{n-2}\,1\,\cdots 1000100101 \mid n \geq 1\,\}$ can be expressed as $\big((0^*1 \cdot A^*) \cap (A^* \cdot 01)\big) \cup \big((0^*1 \cdot A^* \cdot 01) \cap (A^* \cdot 00101)\big)$.

3.

Suppose $L = \{0^n\,1\,0^m\,1\,0^{\max(m,n)} \mid n, m \geq 1\}$ is regular and p the pumping length. Choose $s = 010^p10^p$. According to pumping lemma, there is a partition $s = xyz$ such that $|y| > 0$, $|xy| \leq p$, and for each $i \geq 0$, $xy^iz \in L_A$. Consider the following cases:

i) y contains 1s. Then $xy^0z = xz \notin L$ since any string in L must contain exactly two 1s.

ii) y does not contain 1s. Let $y = 0^k$ where $1 \leq k \leq p$. So there are three possibilities:

1) $x = \epsilon$, $y = 0$, $z = 1\,0^p\,1\,0^p$. Then $xy^{p+1}z = 0^{p+1}\,1\,0^p\,1\,0^p \notin L$.
2) $x = 010^r$, $y = 0^k$, $z = 0^{p-k-r}\,1\,0^p$, where $0 \leq r \leq p-1$. Then $xy^2z = 0\,1\,0^{r+2k+p-k-r}\,1\,0^p = 0\,1\,0^{p+k}\,1\,0^p \notin L$.
3) $x = 010^p10^r$, $y = 0^k$, $z = 0^{p-k-r}$, where $0 \leq r \leq p-1$. Then $xy^2z = 0\,1\,0^p\,1\,0^{r+2k+p-k-r}\,1\,0^p = 0\,1\,0^p\,1\,0^{p+k} \notin L$.

Since one of the above cases must be true, it shows that the pumping lemma cannot be satisfied. Therefore L is not regular.

*Or choose $s = 0^p10^p10^p$ and one only needs to consider the case where $y = 0^+$.

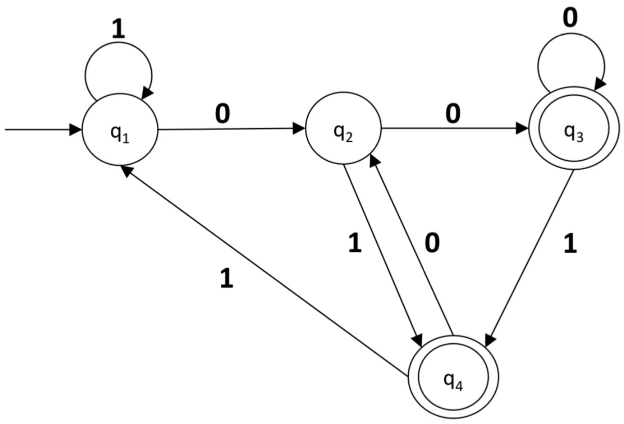
4.

Using the morphism h_1 provided in the hint, we have $h_1^{-1}(L_1) = \{ww \mid w \in \{0, 1, \tilde{0}\}^*\}$. Let $L_a = h_1^{-1}(L_1) \cap 0^*\,1\,\tilde{0}^*\,1 = 0^n\,1\,\tilde{0}^n\,1$, where $n \geq 0$. Consider another morphism $h_2(0) = 0$, $h_2(1) = \epsilon$, $h_2(\tilde{0}) = 1$. Then $h_2(L_a) = 0^n\,\epsilon\,1^n\,\epsilon = 0^n\,1^n$ where $n \geq 0$. So $L_2 = h_2\big(h_1^{-1}(L_1) \cap 0^*\,1\,\tilde{0}^*\,1\big) \cap 0^*011^*$.

5 (a)

There are four equivalence classes of $\equiv_L$:

(1) $[\,11\,] = \{\,\epsilon, 1, 11, 011, \ldots\} = \{\epsilon, 1\} \cup \{$bit string ending with 11$\}$.
(2) $[\,10\,] = \{0, 10, 010, \ldots\} = \{0\} \cup \{$bit string ending with 10$\}$
(3) $[\,00\,] = \{00, 100, 000, \ldots\} = \{0,1\}^*\,00 = \{$bit string ending with 00$\}$.
(4) $[\,01\,] = \{01, 001, 101, \ldots\} = \{0,1\}^*\,01 = \{$bit string ending with 01$\}$.



5 (b)

Since the minimum-sized DFA accepting L has $4 = 2^2 = 2^n$ states, the equivalent NFA must have at least $n + 1 = 3$ states.