Solution to HW1

October 30, 2013

1. Proof by an induction on $|y|$. Base case: $\hat{\delta}(q, x\epsilon) = \hat{\delta}(q, x) = \hat{\delta}(\hat{\delta}(q, x), \epsilon)$. Inductive step: By induction hypothesis we have $\hat{\delta}(q, xy) = \hat{\delta}(\hat{\delta}(q, x), y)$. Then, for any letter a, $\hat{\delta}(q, x(ya)) = \hat{\delta}(q, (xy)a) = \delta(\hat{\delta}(q, xy), a) = \delta(\hat{\delta}(\hat{\delta}(q, x), y), a) = \hat{\delta}(\hat{\delta}(q, x), ya)$.

2. Let A be a DFA accepting R. Since the size of alphabet is 1, out-degree of each state in A must be 1. The reachable part of A must be a path followed by a cycle. For any state q, we define $S_q = \{w|\hat{\delta}(q_0, w) = q\}$, where q_0 is the initial state. It is easy to see that $\{i|0^i \in S_q\}$ is a linear set. Therefore, $\{i|0^i \in R\} = \bigcup_{q \in F}\{i|0^i \in S_q\}$ is semilinear.

3. ($\Leftarrow$) Certainly R is right-invariant. Hence the set must be regular due to Myhill-Nerode Theorem. ($\Rightarrow$) Let A be a DFA accepting that set, then $R_A \subseteq \Sigma^* \times \Sigma^*$: $xR_Ay \Leftrightarrow \forall_{q\in Q}\hat{\delta}(q, x) = \hat{\delta}(q, y)$ is a desired congruence relation.

4. We construct a NFA $N = (2^{Q_\forall \cup Q_\exists}, \Sigma, \delta_N, \{s\}, 2^F \setminus \emptyset)$, where δ_N is defined to be: $\delta_N(X, a) = \{Y \in 2^{Q_\forall \cup Q_\exists} | \bigcup_{q \in X \cap Q_\forall} \delta(q, a) \subseteq Y, \forall_{q \in X \cap Q_\exists} Y \cap \delta(q, a) \neq \emptyset\}$. Let T be a tree representing an accepting run in the AFA, X_i be the set of states in level i of T, and k be the depth of T. It is easy to verify that $X_1 \to X_2 \to \ldots \to X_k$ is also an accepting run in N. On the other hand, given an accepting run $X_1 \to X_2 \to \ldots \to X_k$ in N, we can also construct an accepting run in the AFA. The construction is roughly the reverse of the construction in the opposite direction followed by some trimming of redundant branches.