Skip Lists

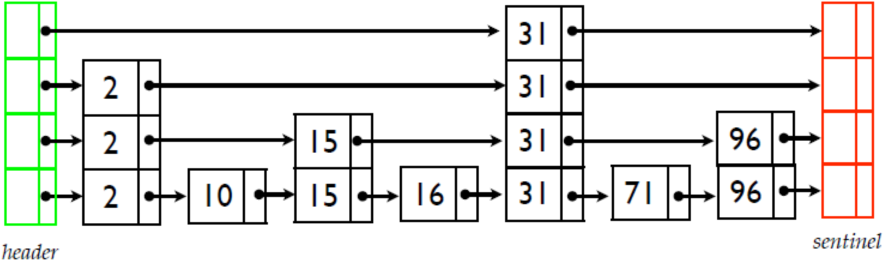
Linked Lists Benefits & Drawbacks

- Benefits:
  - Easy to insert & delete in $O(1)$ time
  - Don't need to estimate total memory needed
- Drawbacks:
  - Hard to search in less than $O(n)$ time (binary search doesn't work, eg.)
  - Hard to jump to the middle
- Skip Lists:
  - fix these drawbacks
  - good data structure for a dictionary ADT

Skip Lists

- Invented around 1990 by Bill Pugh
- Generalization of sorted linked lists V so simple to implement
- Expected search time is $O(\log n)$
- Randomized data structure: use random coin flips to build the data structure

Perfect skip list

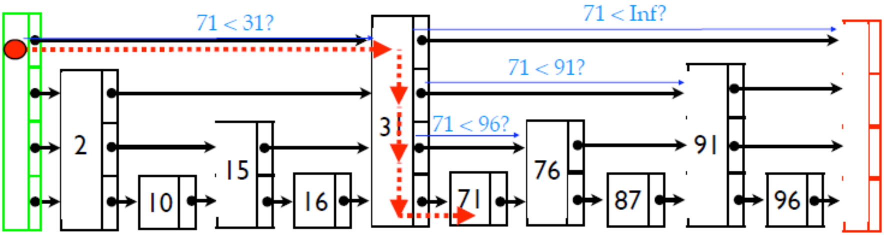


Perfect Skip Lists

- Keys in sorted order.
- $O(\log n)$ levels.
- Each higher level contains $1/2$ the elements of the level below it.
- Header & sentinel nodes are in every level
- Nodes are of variable size: - contain between 1 and $O(\log n)$ pointers
- Pointers point to the start of each node (picture draws pointers horizontally for visual clarity)
- Called skip lists because higher level lists let you skip over many items

Skip Lists

Find 71



When search for *k*:

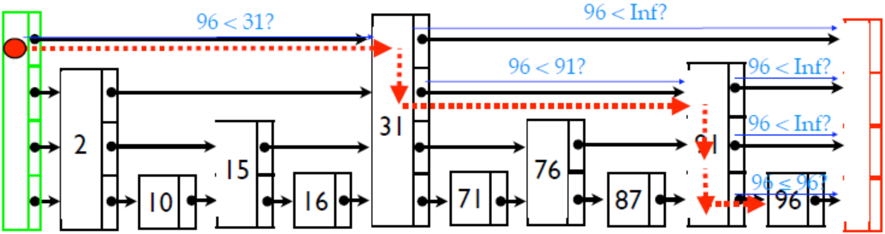
- If k = key, done!
- If k $<$ next key, go down a level
- If k $\geq$ next key, go right

In other words,

- To find an item, we scan along the shortest list until we would ”pass” the desired item.
- At that point, we drop down to a slightly more complete list at one level lower.
- Remember: sorted sequential searching...

```
for(i = 0; i < n; i++)
    if(X[i] >= K) break;
if(X[i] != K) return FAIL;
```

Find 96



When search for *k*:

- If k = key, done!
- If k < next key, go down a level
- If k ≥ next key, go right

Search Time

- $O(\log n)$ levels — because you cut the # of items in half at each level
- Will visit at most 2 nodes per level: If you visit more, then you could have done it on one level higher up.
- Therefore, search time is $O(\log n)$.

Insert & Delete

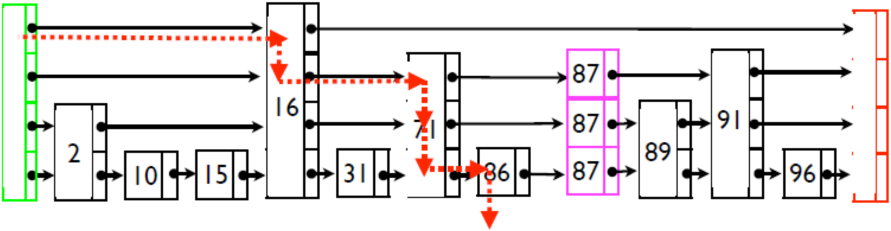
- Insert & delete might need to rearrange the entire list
- Like Perfect Binary Search Trees, Perfect Skip Lists are too structured to support efficient updates.
- Idea:
  - Relax the requirement that each level have exactly half the items of the previous level
  - Instead: design structure so that we expect 1/2 the items to be carried up to the next level
  - Skip Lists are a randomized data structure: the same sequence of inserts/deletes may produce different structures depending on the outcome of random coin flips.

Randomization

- Allows for some imbalance (like the +1 -1 in AVL trees)
- Expected behavior (over the random choices) remains the same as with perfect skip lists.
- Idea: Each node is promoted to the next higher level with probability 1/2
  - Expect 1/2 the nodes at level 1
  - Expect 1/4 the nodes at level 2
  - ...
- Therefore, expect # of nodes at each level is the same as with perfect skip lists.
- Also: expect the promoted nodes will be well distributed across the list

Insertion

Insert 87

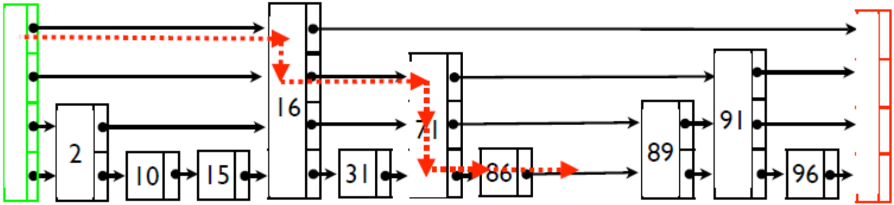



```
Find k
Insert node in level 0
let i = 1
while FLIP() == "heads":
    insert node into level i
    i++
```

Just insertion into a linked list after last visited node in level i

Deletion

Delete 87



There are no ”bad” sequences

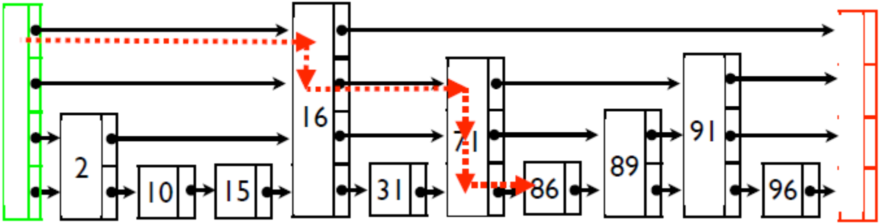
- We expect a randomized skip list to perform about as well as a perfect skip list.
- With some very small probability,
  - the skip list will just be a linked list, or
  - the skip list will have every node at every level
  - These degenerate skip lists are very unlikely!
- Level structure of a skip list is independent of the keys you insert.
- Therefore, there are no ”bad” key sequences that will lead to degenerate skip lists

Skip List Analysis

- Expected number of levels = $O(\log n)$
  - E[# nodes at level 1] = n/2
  - E[# nodes at level 2] = n/4
  - ...
  - E[# nodes at level log n] = 1
- Still need to prove that # of steps at each level is small.

Backwards Analysis

Consider the reverse of the path you took to find k:



Note that you always move up if you can. (because you always enter a node from its topmost level when doing a find)

Analysis, continued...

- What's the probability that you can move up at a give step of the reverse walk?

  0.5

- Steps to go up j levels =
  Make one step, then make either
  - $C(j-1)$ steps if this step went up [Prob = 0.5]
  - $C(j)$ steps if this step went left [Prob = 0.5]
- Expected # of steps to walk up j levels is:

$$C(j) = 1 + 0.5C(j-1) + 0.5C(j)$$

Analysis, continued...

- Expected # of steps to walk up j levels is:

$$C(j) = 1 + 0.5C(j-1) + 0.5C(j)$$

  So:

$$2C(j) = 2 + C(j-1) + C(j)$$
$$C(j) = 2 + C(j-1)$$

  Expected # of steps at each level = 2
- Expanding $C(j)$ above gives us: $C(j) = 2j$
- Since $O(\log n)$ levels, we have $O(\log n)$ steps, expected

Summary

- Skip lists are a randomized data structure
- Provide ”expected” $O(\log(n))$ insert, remove, and search
- Compared to the complexity of the code for structures like an RB-Tree they are fairly easy to implement
- In practice they perform quite well even compared to more complicated structures like balanced BSTs