Amortized Analysis of Splay Trees

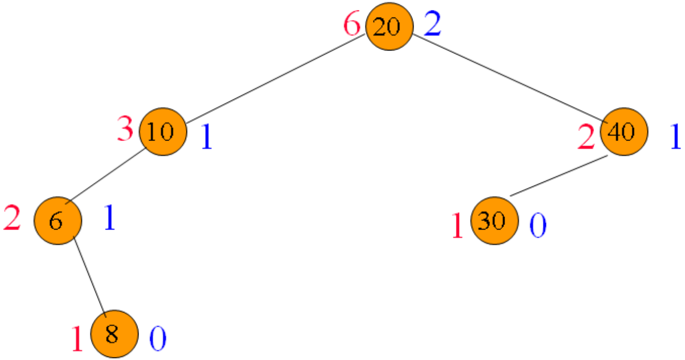
Bottom-Up Splay Trees Analysis

- Amortized complexity of search, insert, delete, and split is $O(\log n)$.
- Actual complexity of each splay tree operation is the same as that of the associated splay.
- Sufficient to show that the amortized complexity of the splay operation is $O(\log n)$.

Potential Function

- *size(x)* = #nodes in subtree whose root is x.
- *rank(x)* = $\lfloor \log_2 \, size(x) \rfloor$. (Recall, e.g., $\lfloor 3.14 \rfloor = 3$.)
- $P(i) = \sum_{tree\ node\ x} \, rank(x)$.
  - $P(i)$ is potential after i'th operation.
  - *size*(x) and *rank(x)* are computed after i'th operation.
  - $P(0) = 0$.
- When join and split operations are done, number of splay trees > 1 at times.
  - $P(i)$ is obtained by summing over all nodes in all trees.

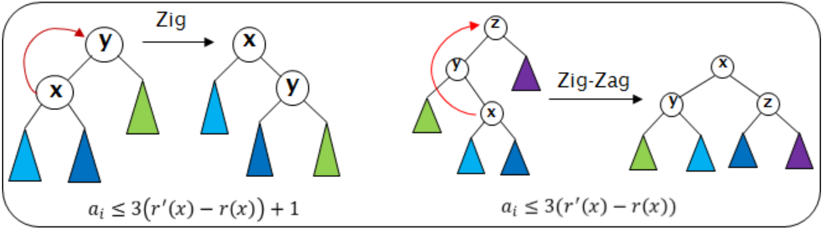
Example



- *size(x)* is in red.
- *rank(x)* is in blue.
- Potential = 5.

We show:
For a splay-step operation on x that transforms the rank function r into r' the amortized cost is:

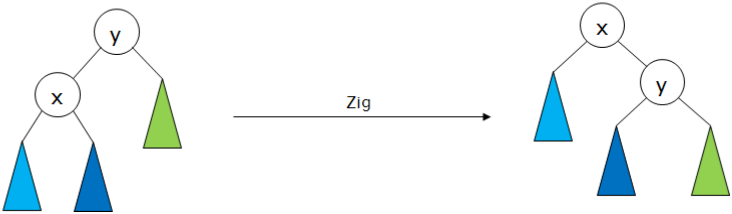
- $a_i \leq 3(r'(x) - r(x)) + 1$ if the parent of x is the root, and
- $a_i \leq 3(r'(x) - r(x))$ otherwise.



Zig (or Zag) Splay Step Amortized Cost

In this case, we have $r'(x) = r(y), r'(y) \leq r'(x)$ and $r'(x) \geq r(x)$. So the amortized cost:

$$
\begin{aligned}
a_i &= 1 + \phi' - \phi \\
&= 1 + r'(x) + r'(y) - r(x) - r(y) \\
&= 1 + r'(y) - r(x) \\
&\leq 1 + r'(x) - r(x) \\
&\leq 1 + 3(r'(x) - r(x))
\end{aligned}
$$



Zig-zig Splay Step Amortized Cost

In this case, we have $r'(x) = r(z), r(y) \geq r(x)$ and $r'(y) \leq r'(x)$.

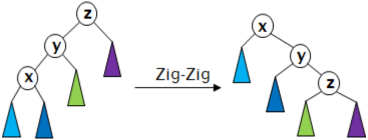
$$\begin{aligned} a_i &= 1 + \phi' - \phi \\ &= 2 + r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z) \\ &= 2 + r'(y) + r'(z) - r(x) - r(y) \\ &\leq 2 + r'(x) + r'(z) - r(x) - r(x) \end{aligned}$$

We use the fact $\frac{1}{2}(\log_2 x + \log_2 y) \leq \log_2 \frac{x+y}{2}$

$\frac{1}{2}(r(x) + r'(z)) \leq \frac{1}{2}(\log_2 s(x) + \log_2 s'(z)) \leq \log_2 \frac{s(x)+s'(z)}{2}$

$\leq \log_2 \frac{s'(x)}{2} = r'(x) - 1 \Rightarrow r'(z) \leq 2r'(x) - r(x) - 2$

Hence, $a_i \leq 2 + r'(x) + r'(z) - r(x) - r(x) \leq 3(r'(x) - r(x))$

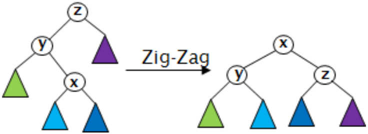


Zig-zag Splay Step Amortized Cost

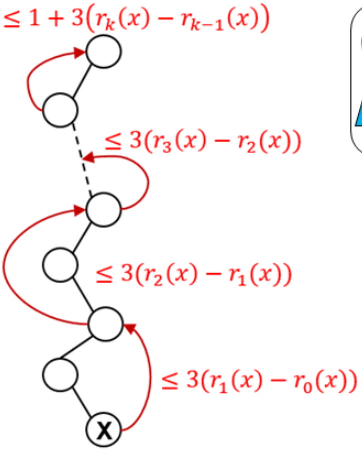
In this case, we have $r'(x) = r(z), r(y) \geq r(x)$.

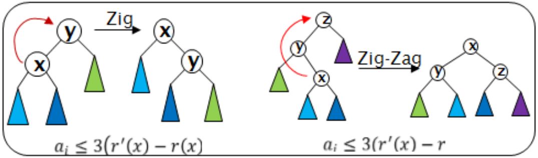
$$
\begin{aligned}
a_i &= 1 + \phi' - \phi \\
&= 2 + r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z) \\
&\leq 2 + r'(y) + r'(z) - r(x) - r(x)
\end{aligned}
$$

$\frac{1}{2}(r'(y) + r'(z)) \leq \frac{1}{2}(\log_2 s'(y) + \log_2 s'(z)) \leq \log_2 \frac{s'(y)+s'(z)}{2}$
$\leq \log_2 \frac{s'(x)}{2} = r'(x) - 1 \Rightarrow r'(y) + r'(z) \leq 2r'(x) - 2$
Hence, $a_i \leq 2 + r'(y) + r'(z) - r(x) - r(x) \leq 3(r'(x) - r(x))$



Splay Operation Amortized Cost





$$
\begin{aligned}
a(splay(x)) &\leq 1 + 3(r_k(x) - \cancel{r_{k-1}(x)}) + 3(\cancel{r_{k-1}(x)} - \cancel{r_{k-2}(x)}) \\
&\quad + \cdots + 3(\cancel{r_1(x)} - r_0(x)) \\
&= 1 + 3(r_k(x) - r_0(x)) \\
&= 1 + 3(r_0(root) - r_0(x)) \\
&= O\left(1 + 3\log\frac{s(root)}{s(x)}\right) \\
&= O(\log n)
\end{aligned}
$$