

# Amortized Analysis of Splay Trees

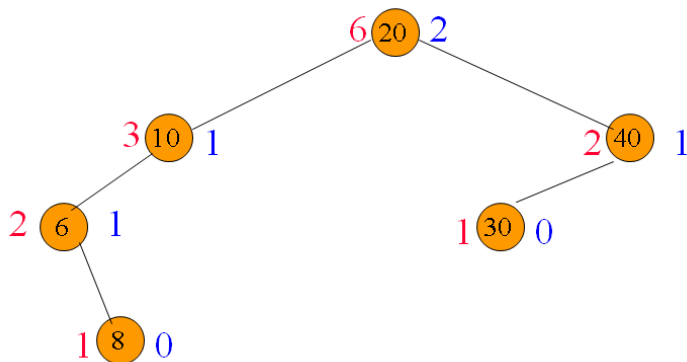
# Bottom-Up Splay Trees Analysis

- Amortized complexity of search, insert, delete, and split is  $O(\log n)$ .
- Actual complexity of each splay tree operation is the same as that of the associated splay.
- Sufficient to show that the amortized complexity of the splay operation is  $O(\log n)$ .

# Potential Function

- $size(x) = \# \text{nodes in subtree whose root is } x$ .
- $rank(x) = \lfloor \log_2 size(x) \rfloor$ . (Recall, e.g.,  $\lfloor 3.14 \rfloor = 3$ .)
- $P(i) = \sum_{\text{tree node } x} rank(x)$ .
  - ▶  $P(i)$  is potential after  $i$ 'th operation.
  - ▶  $size(x)$  and  $rank(x)$  are computed after  $i$ 'th operation.
  - ▶  $P(0) = 0$ .
- When join and split operations are done, number of splay trees  $> 1$  at times.
  - ▶  $P(i)$  is obtained by summing over all nodes in all trees.

# Example



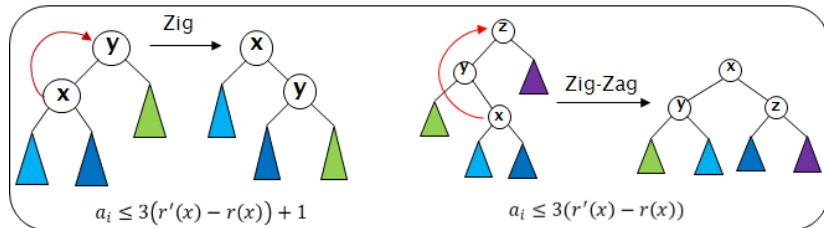
- $size(x)$  is in red.
- $rank(x)$  is in blue.
- Potential = 5.

# Single Splay Step Amortized Analysis

We show:

For a splay-step operation on  $x$  that transforms the rank function  $r$  into  $r'$  the amortized cost is:

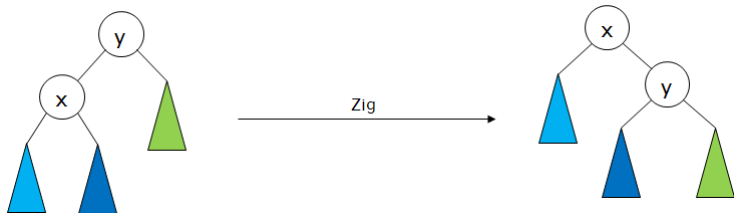
- $a_i \leq 3(r'(x) - r(x)) + 1$  if the parent of  $x$  is the root, and
- $a_i \leq 3(r'(x) - r(x))$  otherwise.



# Zig (or Zag) Splay Step Amortized Cost

In this case, we have  $r'(x) = r(y)$ ,  $r'(y) \leq r'(x)$  and  $r'(x) \geq r(x)$ . So the amortized cost:

$$\begin{aligned} a_i &= 1 + \phi' - \phi \\ &= 1 + r'(x) + r'(y) - r(x) - r(y) \\ &= 1 + r'(y) - r(x) \\ &\leq 1 + r'(x) - r(x) \\ &\leq 1 + 3(r'(x) - r(x)) \end{aligned}$$

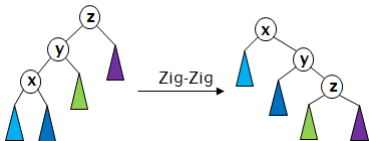


# Zig-zig Splay Step Amortized Cost

In this case, we have  $r'(x) = r(z)$ ,  $r(y) \geq r(x)$  and  $r'(y) \leq r'(x)$ .

$$\begin{aligned}a_i &= 1 + \phi' - \phi \\ &= 2 + r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z) \\ &= 2 + r'(y) + r'(z) - r(x) - r(y) \\ &\leq 2 + r'(x) + r'(z) - r(x) - r(x)\end{aligned}$$

We use the fact  $\frac{1}{2}(\log_2 x + \log_2 y) \leq \log_2 \frac{x+y}{2}$   
 $\frac{1}{2}(r(x) + r'(z)) \leq \frac{1}{2}(\log_2 s(x) + \log_2 s'(z)) \leq \log_2 \frac{s(x)+s'(z)}{2}$   
 $\leq \log_2 \frac{s'(x)}{2} = r'(x) - 1 \Rightarrow r'(z) \leq 2r'(x) - r(x) - 2$   
Hence,  $a_i \leq 2 + r'(x) + r'(z) - r(x) - r(x) \leq 3(r'(x) - r(x))$



# Zig-zag Splay Step Amortized Cost

In this case, we have  $r'(x) = r(z), r(y) \geq r(x)$ .

$$\begin{aligned} a_i &= 1 + \phi' - \phi \\ &= 2 + r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z) \\ &\leq 2 + r'(y) + r'(z) - r(x) - r(x) \end{aligned}$$

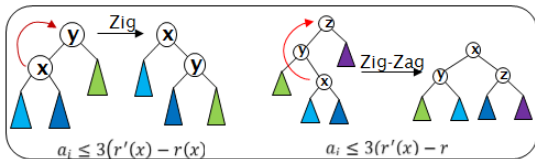
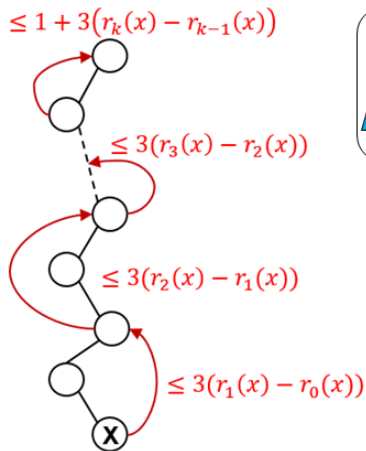
$$\begin{aligned} \frac{1}{2}(r'(y) + r'(z)) &\leq \frac{1}{2}(\log_2 s'(y) + \log_2 s'(z)) \leq \log_2 \frac{s'(y) + s'(z)}{2} \\ &\leq \log_2 \frac{s'(x)}{2} = r'(x) - 1 \Rightarrow r'(y) + r'(z) \leq 2r'(x) - 2 \end{aligned}$$

Hence,  $a_i \leq 2 + r'(y) + r'(z) - r(x) - r(x) \leq 3(r'(x) - r(x))$





# Splay Operation Amortized Cost



$$\begin{aligned}
 a(\text{splay}(x)) &\leq 1 + 3(r_k(x) - r_{k-1}(x)) + 3(r_{k-1}(x) - r_{k-2}(x)) \\
 &\quad + \dots + 3(r_1(x) - r_0(x)) \\
 &= 1 + 3(r_k(x) - r_0(x)) \\
 &= 1 + 3(r_0(\text{root}) - r_0(x)) \\
 &= O\left(1 + 3 \log \frac{s(\text{root})}{s(x)}\right) \\
 &= O(\log n)
 \end{aligned}$$