Amortized Analysis of Splay Trees
Amortized complexity of search, insert, delete, and split is $O(\log n)$.

Actual complexity of each splay tree operation is the same as that of the associated splay.

Sufficient to show that the amortized complexity of the splay operation is $O(\log n)$. 

Potential Function

- $size(x) = \#\text{nodes in subtree whose root is } x$.
- $rank(x) = \lfloor \log_2 size(x) \rfloor$.
- $P(i) = \sum_{\text{tree node } x} rank(x)$.
  - $P(i)$ is potential after $i$’th operation.
  - $size(x)$ and $rank(x)$ are computed after $i$’th operation.
  - $P(0) = 0$.

- When join and split operations are done, number of splay trees $> 1$ at times.
  - $P(i)$ is obtained by summing over all nodes in all trees.
Example

- size(x) is in red.
- rank(x) is in blue.
- Potential = 5.
Splay Step Amortized Cost

- If \( q = \text{null} \) or \( q \) is the root, do nothing (splay is over).
  - \( \Delta P = 0 \)
  - amortized cost = actual cost + \( \Delta P = 0 \).
- If \( q \) is at level 2, do a one-level move and terminate the splay operation.

\[
\begin{align*}
\Delta P &= r'(p) + r'(q) - r(p) - r(q) \\
&\leq r'(q) - r(q)
\end{align*}
\]

- amortized cost = actual cost + \( \Delta P \leq 1 + r'(q) - r(q) \).
2-Level Move (Case 1); Case 2 is similar

- \( r'(q) = r(gp) \leq r'(q) \)
- \( r'(p) \leq r'(q) \leq r(p) \)
- \( \Delta P = r'(gp) + r'(p) + r'(q) - r(gp) - r(p) - r(q) \)
  \( \leq r'(q) + r'(q) - r(q) - r(q) = 2(r'(q) - r(q)) \leq 3(r'(q) - r(q)) - 1 \)
- Amortized cost = actual cost + \( \Delta P \)
  \( \leq 1 + 3(r'(q) - r(q)) - 1 = 3(r'(q) - r(q)) \)
When \( q \neq \text{null} \) and \( q \) is not the root, zero or more 2-level splay steps followed by zero or one 1-level splay step.

Let \( r''(q) \) be rank of \( q \) just after last 2-level splay step.

Let \( r'''(q) \) be rank of \( q \) just after 1-level splay step.

Amortized cost of all 2-level splay steps is \( \leq 3(r''(q) - r(q)) \)

Amortized cost of splay operation
\[
\leq 1 + r''''(q) - r''(q) + 3(r''(q) - r(q)) \\
\leq 1 + 3(r''''(q) - r''(q)) + 3(r''(q) - r(q)) \\
= 1 + 3(r''''(q) - r(q)) \\
\leq 3(\text{floor}(\log_2 n) - r(q)) + 1
\]