Amortized Analysis of Splay Trees

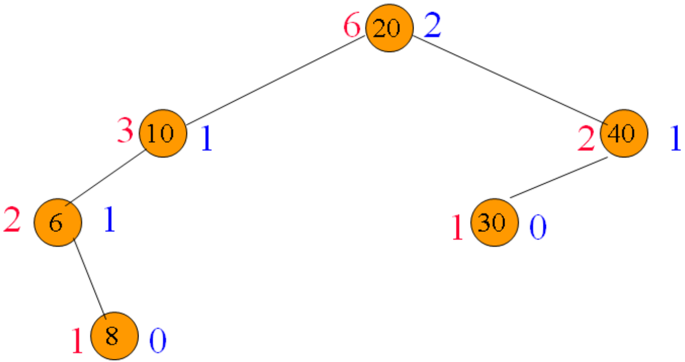
Bottom-Up Splay TreesAnalysis

- Amortized complexity of search, insert, delete, and split is $O(\log n)$.
- Actual complexity of each splay tree operation is the same as that of the associated splay.
- Sufficient to show that the amortized complexity of the splay operation is $O(\log n)$.

Potential Function

- *size(x)* = #nodes in subtree whose root is x.
- *rank(x)* = floor($\log_2$ $size(x)$).
- $P(i) = \sum_{tree\ node\ x} rank(x)$.
  - $P(i)$ is potential after i'th operation.
  - $size(x)$ and *rank(x)* are computed after i'th operation.
  - $P(0) = 0$.
- When join and split operations are done, number of splay trees > 1 at times.
  - $P(i)$ is obtained by summing over all nodes in all trees.

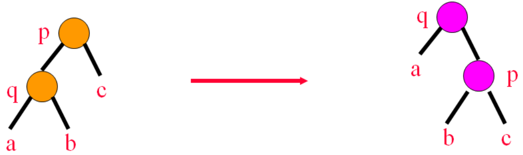
Example



- *size(x)* is in red.
- *rank(x)* is in blue.
- Potential = 5.

Splay Step Amortized Cost

- If q = null or q is the root, do nothing (splay is over).
  - $\Delta P = 0$
  - amortized cost = actual cost + ΔP = 0.
- If q is at level 2, do a one-level move and terminate the splay operation.



  - $r(x)$ = rank of x before splay step.
  - $r'(x)$ = rank of x after splay step.
  - $\Delta P = r'(p) + r'(q) - r(p) - r(q) \leq r'(q) - r(q)$
  - amortized cost = actual cost + $\Delta P \leq 1 + r'(q) - r(q)$.

2-Level Move (Case 1); Case 2 is similar

- $r'(q) = r(gp) \qquad r'(gp) \leq r'(q)$
  $r'(p) \leq r'(q) \qquad r(q) \leq r(p)$
- $\Delta P = r'(gp) + r'(p) + r'(q) - r(gp) - r(p) - r(q)$
  $\leq r'(q) + r'(q) - r(q) - r(q) = 2(r'(q) - r(q)) \leq 3(r'(q) - r(q)) - 1$
- amortized cost = actual cost + ΔP
  $\leq 1 + 3(r'(q) - r(q)) - 1 = 3(r'(q) - r(q))$

Splay Operation

- When $q \neq$ null and q is not the root, zero or more 2-level splay steps followed by zero or one 1-level splay step.
- Let $r''(q)$ be rank of q just after last 2-level splay step.
- Let $r'''(q)$ be rank of q just after 1-level splay step
- Amortized cost of all 2-level splay steps is $\leq 3(r''(q) - r(q))$
- Amortized cost of splay operation
  $\leq 1 + r'''(q) - r''(q) + 3(r''(q) - r(q))$
  $\leq 1 + 3(r'''(q) - r''(q)) + 3(r''(q) - r(q))$
  $= 1 + 3(r'''(q) - r(q))$
  $\leq 3(floor(\log_2 n) - r(q)) + 1$