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Optimal Wire-Sizing Function Under the Elmore Delay Model With Bounded Wire Sizes

Yu-Min Lee, Charlie Chung-Ping Chen, and D. F. Wong

***Abstract*—In this brief, we develop the optimal wire-sizing functions under the Elmore delay model with bounded wire sizes. Given a wire segment of length $\mathcal{L}$, let $f(x)$ be the width of the wire at position x, $0 \leq x \leq \mathcal{L}$. We show that the optimal wire-sizing function that minimizes the Elmore delay through the wire is $f(x) = ae^{-bx}$, where $a > 0$ and $b > 0$ are constants that can be computed in $O(1)$ time. In the case where lower bound ($L > 0$) and upper bound ($U > 0$) of the wire widths are given, we show that the optimal wire-sizing function $f(x)$ is a truncated version of ae^{-bx} that can also be determined in $O(1)$ time. Our wire-sizing formula can be iteratively applied to optimally size the wire segments in a routing tree.**

***Index Terms*—Elmore delay, optimal, wire sizing.**

I. INTRODUCTION

As very large-scale integration (VLSI) technology continues to scale down, interconnect delay has become the dominant factor in deep submicron designs. As a result, wire sizing plays an important role in achieving desirable circuit performance. Recently, many wire-sizing algorithms have been reported in the literature [1]–[5]. All these algorithms size each wire segment uniformly, i.e., identical width at every position on the wire. In order to achieve nonuniform wire sizing, existing algorithms have to chop wire segments into large number of small segments. Consequently, the number of variables in the optimization problem is increased substantially and thus results in long runtime and large storage.

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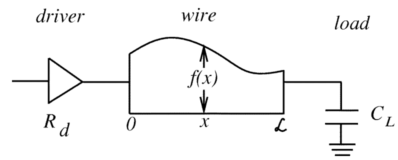

Fig. 1. Six types of optimal wire-sizing functions.

In [6], the optimal wire shape with minimal Elmore delay without wire-size constraints are presented using the calculus of variation methods. In this brief, we develop the optimal wire-sizing function for minimal Elmore delay with the wire-size constraints using only basic mathematical methods. Given a wire segment W of length $\mathcal{L}$, a source with driver resistance R_d, and a sink with load capacitance C_L. For each $x \in [0, \mathcal{L}]$, let $f(x)$ be the wire width of W at position x. Fig. 1 shows an example. Let r_0 and c_0 be the respective wire resistance and wire capacitance per unit square. Let D be the Elmore delay from the source to the sink of W. We show that the optimal wire-sizing function f that minimizes D satisfies a differential equation which can be analytically solved. We have $f(x) = ae^{-bx}$, where $a > 0$ and $b > 0$ are constants that can be computed in $O(1)$ time. These constants depend on R_d, C_L, $\mathcal{L}$, r_0, and c_0. Our method is extended to solve the case where lower bound ($L > 0$) and upper bound ($U > 0$) on the wire widths are given, i.e., $L \leq f(x) \leq U$, $0 \leq x \leq \mathcal{L}$, we show that the optimal wire-sizing function $f(x)$ is a truncated version of ae^{-bx} which can also be determined in $O(1)$ time. Our wire-sizing formula can be iteratively applied to optimally size the wire segments in a routing tree.

The remainder of this brief is organized as follows. In Section II, we show how to compute the Elmore delay for nonuniformly sized wire segments. In Section III-A, we derive the optimal wire-sizing function when the wire widths are not constrained by any bounds. In Section III-B, we consider the case where lower and upper bounds for the wire widths are given. We discuss the importance of our wire-sizing formula in sizing the wire segments in a routing tree in Section IV. Finally, we present some experimental results and concluding remarks in Section V.

II. ELMORE DELAY MODEL

We use the Elmore delay model [7]. Suppose W is partitioned into n equal-length wire segments, each of length $\triangle x = \mathcal{L}/n$. Let x_i be $i\triangle x$, $1 \leq i \leq n$. The capacitance and resistance of a wire segment i can be approximated by $c_0 \triangle x f(x_i)$ and $r_0 \triangle x / f(x_i)$, respectively. Thus, the Elmore delay through W can be approximated by

$$D_n = R_d \left(C_L + \sum_{i=1}^{n} c_0 f(x_i) \triangle x \right) + \sum_{i=1}^{n} \frac{r_0 \triangle x}{f(x_i)} \left(\sum_{j=i}^{n} c_0 f(x_j) \triangle x + C_L \right). \quad (1)$$

The first term is the delay of the driver, which is given by the driver resistance R_d multiplied by the total capacitance of W and C_L. The second term is the sum of the delay in each wire segment i, which is given by its own resistance $r_0 \triangle x / f(x_i)$ multiplied by its downstream capacitance $\sum_{j=i}^{n} c_0 f(x_j) \triangle x + C_L$. (See Fig. 2.) As $n \to \infty$, $D_n \to D$ where

$$D = R_d \left(C_L + \int_0^{\mathcal{L}} c_0 f(x)\, dx \right) + \int_0^{\mathcal{L}} \frac{r_0}{f(x)} \left(\int_x^{\mathcal{L}} c_0 f(t)\, dt + C_L \right) dx \quad (2)$$

is the Elmore delay through the driver and W.

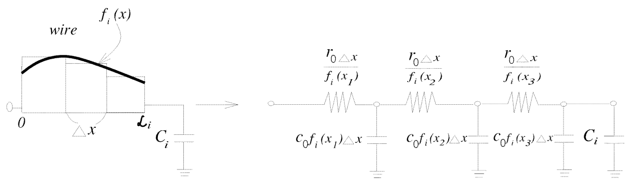

Fig. 2. Nonuniform wire sizing.

III. Optimal Wire-Sizing Function

In this section, we derive closed-form formula for the optimal wire-sizing function. It is reasonable to assume that wire-sizing functions are bounded and piecewise smooth with at most finite number of discontinuity points. We consider two cases *unconstrained* and *constrained wire sizing*. In unconstrained wire sizing, there is no bound on the value of $f(x)$; i.e., we determine $f\colon [0, \mathcal{L}] \rightarrow (0, \infty)$ that minimizes D. In constrained wire-sizing, we are given $L > 0$ and $U < \infty$, and require that $L \leq f(x) \leq U$, $0 \leq x \leq \mathcal{L}$; i.e., we determine $f\colon [0, \mathcal{L}] \rightarrow [L, U]$ that minimizes D.

A. Unconstrained Wire Sizing

We now consider unconstrained wire sizing. We show that the optimal wire-sizing function satisfies a second-order ordinary differential equation which can be analytically solved.

Theorem 1: Let f be an optimal wire-sizing function. We have

$$f^2(x) = \frac{r_0\left(C_L + c_0\int_x^{\mathcal{L}} f(t)\,dt\right)}{c_0\left(R_d + r_0\int_0^x \frac{1}{f(t)}\,dt\right)}. \tag{3}$$

Proof: Let $x \in [0, \mathcal{L}]$. Assume f is continuous at x. We consider $\hat{f}$ which is a local modification of f in a small region $[x-(\delta/2),\, x+(\delta/2)]$. The function $\hat{f}$ is defined as follows:

$$\hat{f}(t) = \begin{cases} y, & x - \dfrac{\delta}{2} \leq t \leq x + \dfrac{\delta}{2} \\ f(t), & \text{otherwise.} \end{cases} \tag{4}$$

The wire W could be divided into three regions Ω_1, Ω_2, and Ω_3 as shown in Fig. 3. We denote the signal delay through Ω_i by D_i. Hence the total signal delay $D = \sum_{i=1}^{3} D_i$. We represent the wire resistance (capacitance) of Ω_i by R_i (C_i). We have $R_2 = r_0\delta/y$ and $C_2 = c_0\delta y$. The signal delay through the wire can be calculated as follows:

$$\begin{aligned} D = {} & R_d(C_L + C_1 + c_0\delta y + C_3 + C_L) + \int_0^{x-(\delta/2)} \frac{r_0}{f(t)} \\ & \cdot \left(\int_t^{x-(\delta/2)} c_0 f(s)\,ds + c_0\delta y + C_3 + C_L\right) dt \\ & + \frac{r_0\delta}{y}\,(c_0\delta y + C_3 + C_L) \\ & + \int_{x+(\delta/2)}^{\mathcal{L}} \frac{r_0}{f(t)} \left(\int_t^{\mathcal{L}} c_0 f(s)\,ds + C_L\right) dt. \end{aligned} \tag{5}$$

Thus

$$\frac{dD}{dy} = c_0\delta\left(R_d + r_0\int_0^{x-(\delta/2)} \frac{1}{f(t)}\,dt\right) - \frac{r_0\delta\left(C_L + c_0\int_{x+(\delta/2)}^{\mathcal{L}} f(t)\,dt\right)}{y^2}. \tag{6}$$

By setting $dD/dy = 0$, we get

$$y_{\min}^2 = \frac{r_0\left(C_L + c_0\int_{x-(\delta/2)}^{\mathcal{L}} f(t)\,dt\right)}{c_0\left(R_d + r_0\int_0^{x+(\delta/2)} \frac{1}{f(t)}\,dt\right)}. \tag{7}$$

Therefore, $\hat{f}$ using $y = y_{\min}$ gives minimum delay.

Let $\delta \rightarrow 0$, we get

$$y_{\min}^2 = \frac{r_0\left(C_L + c_0\int_x^{\mathcal{L}} f(t)\,dt\right)}{c_0\left(R_d + r_0\int_0^x \frac{1}{f(t)}\,dt\right)}. \tag{8}$$

Since f is an optimal wire-sizing function, we have $y_{\min} = f(x)$, and hence

$$f^2(x) = \frac{r_0\left(C_L + c_0\int_x^{\mathcal{L}} f(t)\,dt\right)}{c_0\left(R_d + r_0\int_0^x \frac{1}{f(t)}\,dt\right)}. \tag{9}$$

For the case where f is not continuous at x, we have f is either left-continuous or right-continuous at x. All we need to do is to start with using the interval $[x-\delta,\, x]$ or $[x,\, x+\delta]$, respectively. □

Note that $C_L + c_0\int_x^{\mathcal{L}} f(t)\,dt$ is equal to the downstream capacitance at point x [denoted by $\Gamma(x)$] and $R_d + r_0\int_0^x (1/f(t))\,dt$ is equal to the upstream resistance at point x [denoted by $\Phi(x)$]. Hence, we can rewrite (3) as

$$f(x) = \sqrt{\frac{r_0\Gamma(x)}{c_0\Phi(x)}}. \tag{10}$$

Since Γ is strictly decreasing and Φ is strictly increasing, therefore f is strictly decreasing.

By rearranging the terms in (3) and differentiating it with respect to x twice, we get the following theorem.

Theorem 2: Let $f(x)$ be an optimal wire-sizing function. We have

$$f''(x)f(x) = f'(x)^2. \tag{11}$$

Proof: We first multiply (3) by the denominator of its right hand side and then differentiating both side with respect to x. We get

$$2f(x)f'(x)\left(R_d + r_0\int_0^x \frac{1}{f(t)}\,dt\right) = -2r_0 f(x). \tag{12}$$

Since $f(x) \neq 0$, we can divide both side by $f(x)$ and get

$$f'(x)\left(R_d + r_0\int_0^x \frac{1}{f(t)}\,dt\right) = -r_0. \tag{13}$$

Since f is strictly decreasing, $f'(x) < 0$. Dividing the above equation by $f'(x)$ and then differentiate both sides with respect to x, we obtain

$$f''(x)f(x) = f'(x)^2. \tag{14}$$

□

We can analytically solve the differential equation (11) and obtain a closed-form solution. We have the following theorem.

Theorem 3: Let $f(x) = ae^{-bx}$, where $a = r_0/bR_d$ and

$$b\sqrt{\frac{R_dC_L}{r_0c_0}} - e^{(-b\mathcal{L})/2} = 0. \tag{15}$$

We have that f is an optimal wire-sizing function.

Proof: Let $y = f(x)$ and $P = y'$. We have $y'' = P(dP/dy)$. The differential equation (11) can be rewritten as

$$P\left(y\,\frac{dP}{dy} - P\right) = 0. \tag{16}$$

Since $P = f'(x) < 0$, we have

$$y\,\frac{dP}{dy} - P = 0. \tag{17}$$

Separating P and y, we get

$$\frac{dP}{P} = \frac{dy}{y}. \tag{18}$$

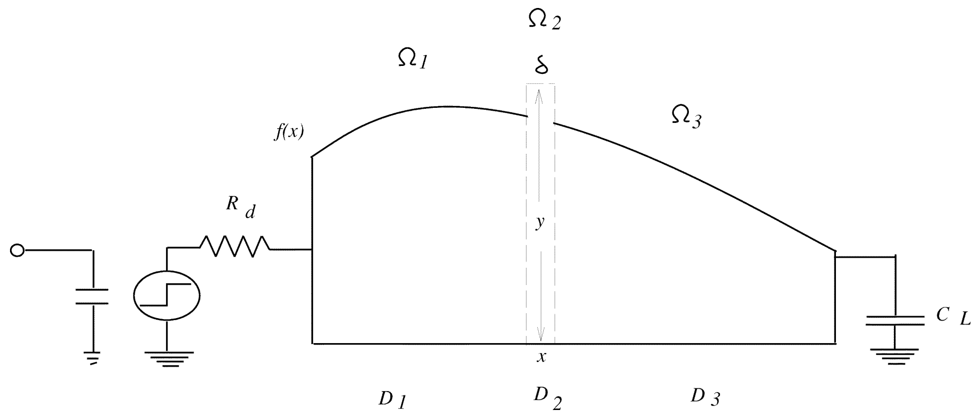

Fig. 3. Elmore delay model.

Integrating both sides, we get

$$P = c_1 y \tag{19}$$

where c_1 is a constant. Since $P = y'$, we have

$$\frac{dy}{dx} = c_1 y. \tag{20}$$

Separating the variables and integrating both sides, we get

$$c_1 x + c_2 = \ln y \tag{21}$$

where c_2 is a constant. It follows that

$$y = e^{c_1 x + c_2} = ae^{-bx} \tag{22}$$

where $a > 0$ and $b > 0$. (Note that $b > 0$ follows from the fact that f is decreasing.)

In order to determine a and b, we substitute $f(x) = ae^{-bx}$ into (9) and check the two boundary points $x = 0$ and $x = \mathcal{L}$. We obtain the following:

$$c_0 b R_d a^2 + r_0 c_0 \left(e^{-b\mathcal{L}} - 1\right) a - r_0 b C_L = 0 \tag{23}$$

$$c_0 b R_d a^2 + r_0 c_0 \left(e^{b\mathcal{L}} - 1\right) a - r_0 b C_L e^{2b\mathcal{L}} = 0. \tag{24}$$

We can simplify these two equations and get

$$ab = \frac{r_0}{R_d} \tag{25}$$

$$b \sqrt[2]{\frac{R_d C_L}{r_0 c_0}} - e^{(-b\mathcal{L})/2} = 0. \tag{26}$$

□

Note that the function $g(z) = z\sqrt{(R_d C_L / r_0 c_0)} - e^{(-z\mathcal{L})/2}$ is a strictly increasing function in z, $g(0) < 0$, and $\lim_{z\to\infty} g(z) > 0$. Thus $g(z)$ has a unique root $b > 0$. We can use Newton–Raphson method [8] to determine b and, in practice, five to seven iterations are sufficient. Since $a = r_0/R_d b$ and $b > 0$, we have $a > 0$. Fig. 4 shows the exponentially decreasing nature of the optimal wire-sizing function.

B. Constrained Wire-Sizing

We now consider constrained wire sizing. It is clear that if the wire-sizing function f obtained for the unconstrained case lies within bounds L and U, then f is also optimal for constrained wire sizing. On the other hand, if for some x, $f(x)$ is not in $[L, U]$, a simple approach is to round $f(x)$ to either L or U; i.e., the new function is obtained by a direct truncation of f by $y = L$ and $y = U$. (See Fig. 5.) Unfortunately, the resulting function is not optimal. The reason is as follows: Suppose the curves $f(x) = ae^{-bx}$ and $y = U$ intersect at $x = v$, from (10) v must satisfy

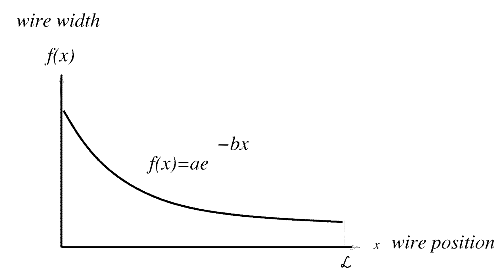

Fig. 4. Local modification of an optimal wire-sizing function.

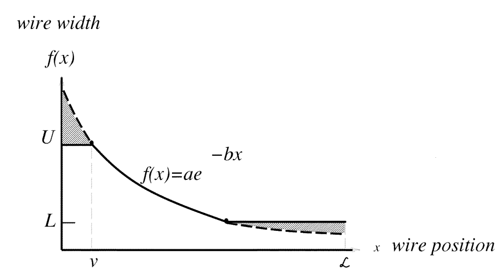

Fig. 5. Optimal unconstrained wire sizing.

$$f(v) = \sqrt{\frac{r_0 \Gamma(v)}{c_0 \Phi(v)}} \tag{27}$$

for v to be on the optimal curve. However, from Fig. 5, it is clear that v does not satisfy (27), because both of its upstream resistance and downstream capacitance should be recalculated according to the new function, in which the two values associated with v are reduced because of the truncation. Thus, this simple approach is not optimal.

Recall that the optimal unconstrained wire-sizing function is a decreasing function. We can show that the optimal constrained wire-sizing function must also be decreasing.

Theorem 4: Let f be an optimal constrained wire-sizing function. We have, f is decreasing on $[0, \mathcal{L}]$.

According to Theorem 4, the optimal wire-sizing function f, similar to the one shown in Fig. 5, consists of (at most) three parts. The first part is $f(x) = U$, the middle part is a decreasing function, and the last part is $f(x) = L$. The three parts of $f(x)$ partition W into three wire segments, A, B, and C, where A has width U, C has width L, and B is defined by the middle part of $f(x)$. It is easy to see that the middle part of $f(x)$ must be of the form $f(x) = ae^{-bx}$ for some $a > 0$ and

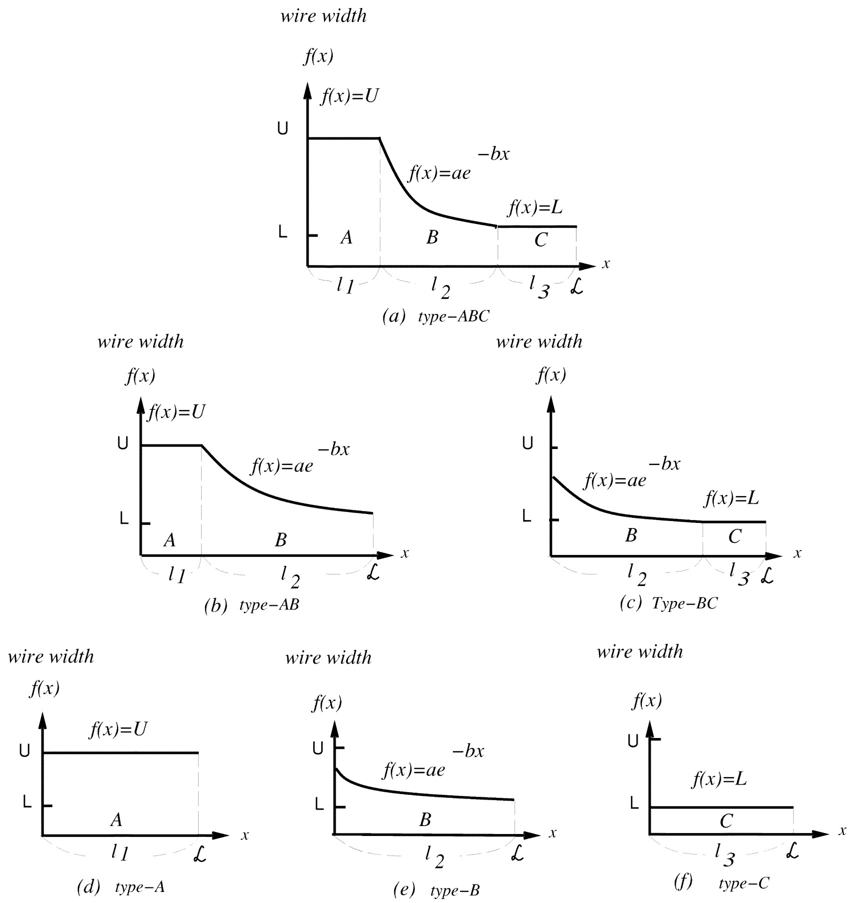

Fig. 6. Direct truncation is not optimal.

$b > 0$. To see this, we can consider the wire segment A to be a part of the driver and its resistance to be a part of R_d. Similarly, the wire segment C can be considered as a part of the load and its capacitance as a part of C_L. According to (15), we can re-calculate a and b using the new values of R_d and C_L, as long as we know the length of the wire segments A and B.

As mentioned before, not all three parts of $f(x)$ needed to be present. In fact, an optimal constrained wire-sizing function $f(x)$ can be of any one of the six types of functions (type-A, type-B, type-C, type-AB, type-BC, and type-ABC) as shown in Fig. 6. Note that the six function types clearly are named after the wire-segment types which are presented in W. For example, in a type-AB function, W consists only of wire segments A and B. As shown in Fig. 6, l_1, l_2, and l_3 are the length of wire segments A, B, and C, respectively.

We now define six wire-sizing functions f_A, f_B, f_C, f_{AB}, f_{BC}, f_{ABC} as follows: All six functions are of the form

$$f(x) = \begin{cases} U, & 0 \le x \le l_1, \\ ae^{-b(x-l_1)}, & l_1 \le x \le l_1 + l_2, \\ L, & l_1 + l_2 \le x \le l_1 + l_2 + l_3 = \mathcal{L} \end{cases} \tag{28}$$

where the parameters a, b, l_1, l_2, and l_3 for the six functions are given in Table I. Typically, the names of the functions correspond to their types, i.e., f_A is of type-A, f_B is of type-B, and so on, but it is not always true. For example, it is possible that after we compute the parameters for f_{AB} we get $l_1 \ge \mathcal{L}$ and hence it is of type-A; it is also possible that f_{AB} degenerates into a type-B function. In this case, we say that f_{AB} is *degenerated*. We also note that sometimes the functions may be *illegal* in the sense that they violate the wire-width constraints. Nevertheless, we can show that these six functions are candidates for an optimal constrained wire-sizing function $f(x)$. In fact, if we eliminate the functions that are either illegal or degenerated, an optimal wire-sizing function can be chosen as the best one (in terms of delay) among the remaining ones. We have the following theorem.

Theorem 5: Let $G \subseteq F = \{f_A, f_B, f_C, f_{AB}, f_{BC}, f_{ABC}\}$ be the set of functions that are either illegal or degenerated. Let $f \in F - G$ be a function which has minimum delay. We have, f is an optimal constrained wire-sizing function.

The above method always requires the computation of all six functions in F. With the help of additional analysis, we can speed up the procedure. Table II shows a set of six *feasibility conditions* $\{\varphi_A, \varphi_B, \varphi_C, \varphi_{AB}, \varphi_{BC}, \varphi_{ABC}\}$ on $\mathcal{L}$. Let $\Gamma = \{A, B, C, AB, CB, ABC\}$.

Lemma 1: The six feasibility conditions $\{\varphi_A, \varphi_B, \varphi_C, \varphi_{AB}, \varphi_{BC}, \varphi_{ABC}\}$ cover all possible $\mathcal{L} > 0$. Moreover, if $\mathcal{L}$ satisfies φ_z, where $z \in \Gamma$, then f_z is legal and is of type-z.

Theorem 6: Let $H = \{f_z | z \in \Gamma$ and $\mathcal{L}$ satisfies $\varphi_z\}$. Let $f \in H$ be a function which has minimum delay. We have, f is an optimal constrained wire-sizing function.

According to Theorem 6, we only need to check the six feasibility conditions. Only the functions in H needed to be computed. In general, $|H| < 6$ and we have never encountered any case where $|H| \neq 1$.

We also have the following interesting observations. In Fig. 7, we show the relationships among the six types of optimal wire-sizing functions with respect to the three parameters: wire length $\mathcal{L}$, driver resistance R_d, and load capacitance C_L. The horizontal axis represents the trend of the driver resistance and the load capacitance. The vertical axis

TABLE I
DEFINITIONS OF THE WIRE-SIZING FUNCTION $f_A, f_B, f_C, f_{AB}, f_{BC}$, AND f_{ABC}

	l_1	l_2	l_3	a	b
f_A	$\mathcal{L}$	0	0	U	0
f_B	0	$\mathcal{L}$	0	$\Psi_1(a) = 0$	$\frac{r_0}{aR_d}$
f_C	0	0	$\mathcal{L}$	L	0
f_{AB}	$\Psi_2(l_1) = 0$	$\mathcal{L} - l_1$	0	U	$\frac{r_0}{R_dU+r_0l_1}$
f_{BC}	0	$\mathcal{L} - l_3$	$\Psi_3(l_3) = 0$	$\frac{r_0(C_L+c_0Ll_3)}{R_dc_0L}$	$\frac{c_0L}{C_L+c_0Ll_3}$
f_{ABC}	$\frac{\frac{C_L}{c_0L}+\mathcal{L}-(1+ln\frac{U}{L})\frac{UR_d}{r_0}}{2+ln\frac{U}{L}}$	$\frac{ln\frac{U}{L}(\frac{C_L}{c_0L}+\mathcal{L}+\frac{UR_d}{r_0})}{2+ln\frac{U}{L}}$	$\mathcal{L} - l_2 - l_1$	U	$\frac{r_0}{R_dU+r_0l_1}$

$\Psi_1(a) = a^2 - \frac{r_0C_L}{c_0R_d}e^{\frac{r_0\mathcal{L}}{aR_d}}$; $\Psi_2(l_1) = \frac{r_0\mathcal{L}+R_dU}{R_dU+r_0l_1} - ln\frac{c_0U(R_dU+r_0l_1)}{r_0C_L} - 1$

$\Psi_3(l_3) = \frac{C_L+c_0L\mathcal{L}}{C_L+c_0Ll_3} - ln\frac{r_0(C_L+c_0Ll_3)}{c_0R_dL^2} - 1$

TABLE II
FEASIBILITY CONDITIONS

φ_A	$\mathcal{L} \leq \frac{C_L}{c_0U} - \frac{R_dU}{r_0}$
φ_B	$\mathcal{L} \leq min\{\frac{UR_d}{r_0}ln\frac{c_0R_dU^2}{r_0C_L}\ ,\ \frac{C_L}{r_0L}ln\frac{r_0C_L}{c_0R_dL^2}\}$
φ_C	$\mathcal{L} \leq \frac{R_dL}{r_0} - \frac{C_L}{c_0L}$
φ_{AB}	$\mathcal{L} > max\{\frac{UR_d}{r_0}ln\frac{c_0R_dU^2}{r_0C_L}\ ,\ \frac{C_L}{c_0U} - \frac{R_dU}{r_0}\}$ and $\mathcal{L} \leq (1+ln\frac{U}{L})\frac{C_L}{c_0L} - \frac{R_dU}{r_0}$
φ_{BC}	$\mathcal{L} > max\{\frac{R_dL}{r_0} - \frac{C_L}{c_0L}\ ,\ \frac{C_L}{r_0L}ln\frac{r_0C_L}{c_0R_dL^2}\}$ and $\mathcal{L} \leq (1+ln\frac{U}{L})\frac{R_dU}{r_0} - \frac{C_L}{c_0L}$
φ_{ABC}	$\mathcal{L} > max\{(1+ln\frac{U}{L})\frac{R_dU}{r_0} - \frac{C_L}{c_0L}\ ,\ (1+ln\frac{U}{L})\frac{C_L}{c_0L} - \frac{R_dU}{r_0}\}$

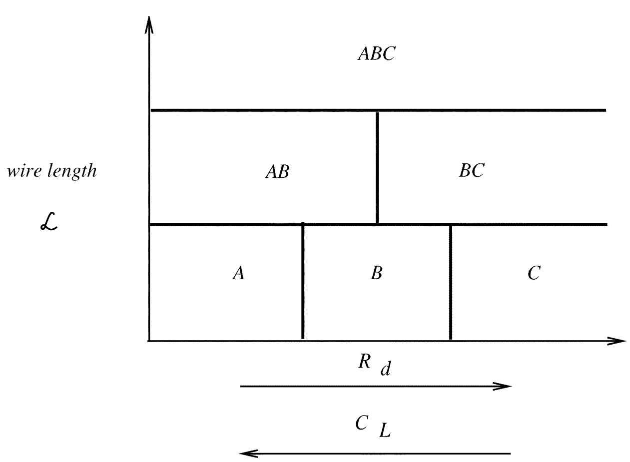

Fig. 7. Relationships among the six types of functions with respect to $\mathcal{L}$ and R_d and C_L.

TABLE III
RC PARAMETERS

Unit Resistance:	0.008 $\Omega/\mu m$
Unit Capacitance:	$6.0 * 10^{-17}$ $F/\mu m$
Minimum Wire Width:	1.0 μm
Maximum Wire Width:	3.5 μm
Driver Resistance:	25 Ω
Load Capacitance:	$1.0 * 10^{-12}$ F

represents the wire length $\mathcal{L}$. Suppose we keep R_d and C_L fixed and varies $\mathcal{L}$. When $\mathcal{L}$ is small, optimal wire-sizing functions tend to be of type-A, type-B, or type-C. As we increase $\mathcal{L}$, wire-sizing function types will change to type-AB or type-BC when $\mathcal{L}$ is of moderate size and will be of type-ABC when $\mathcal{L}$ is large. Suppose we keep $\mathcal{L}$ fixed and varies R_d and C_L. When $\mathcal{L}$ is small, as we increase R_d or decrease C_L, optimal wire-sizing function will change from type-A to type-B and then to type-C. When $\mathcal{L}$ is of moderate size, optimal wire-sizing function will change from type-AB to type-BC as we increase R_d or decrease C_L. Roughly speaking, the larger the R_d or $1/C_L$, the smaller the wire sizes. When the wire length $\mathcal{L}$ is very large, the optimal wire-sizing function is most likely to be of type-ABC.

TABLE IV
NUMBER OF NEWTON–RAPHSON ITERATIONS

Precision Requirement (μm)	# of iterations
0.1	5
0.01	5
0.001	5
0.0001	6
0.00001	6
0.000001	7

IV. APPLICATION TO ROUTING TREES

Our wire-sizing formula can be applied to size a general routing tree. Recently, [2] presents a wire-sizing algorithm GWSA-C for continuously sizing the wire segments in routing trees to minimize weighted delay. Each segment in the tree is sized uniformly, i.e. uniform wire width per segment. Basically, GWSA-C is an iterative algorithm with guaranteed convergence to a global optimal solution. In each iteration of GWSA-C, the wire segments are examined one at a time; each time a wire segment is uniformly re-sized optimally while keeping the widths of the other segments fixed. We can incorporate our wire-sizing formula into GWSA-C to size each wire segment nonuniformly. When we apply our wire-sizing formula to size a wire segment in a tree, R_d should be set to be the total (weighted) upstream resistance including the driving resistance, and the C_L should be set to be the total (weighted) downstream capacitance, including the load capacitances of the sinks in the subtree. (See Fig. 8.) It can be shown that this modified algorithm is extremely fast and always converges to a global optimal solution.

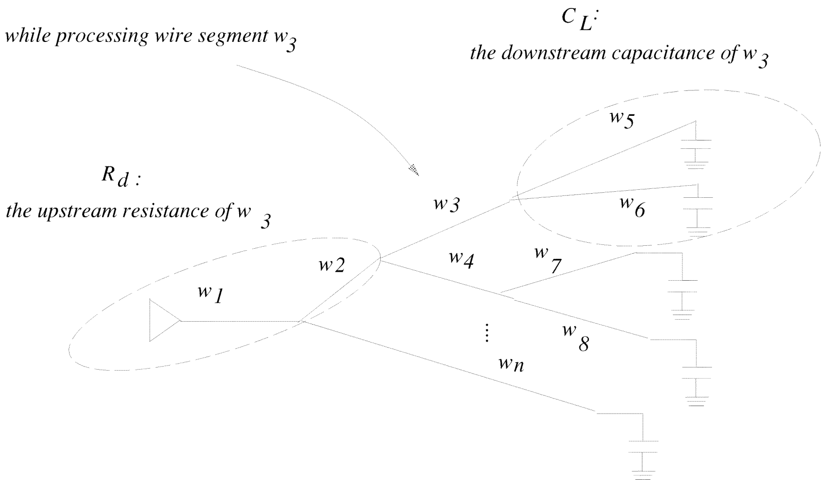

Fig. 8. Sizing a segment of a routing tree.

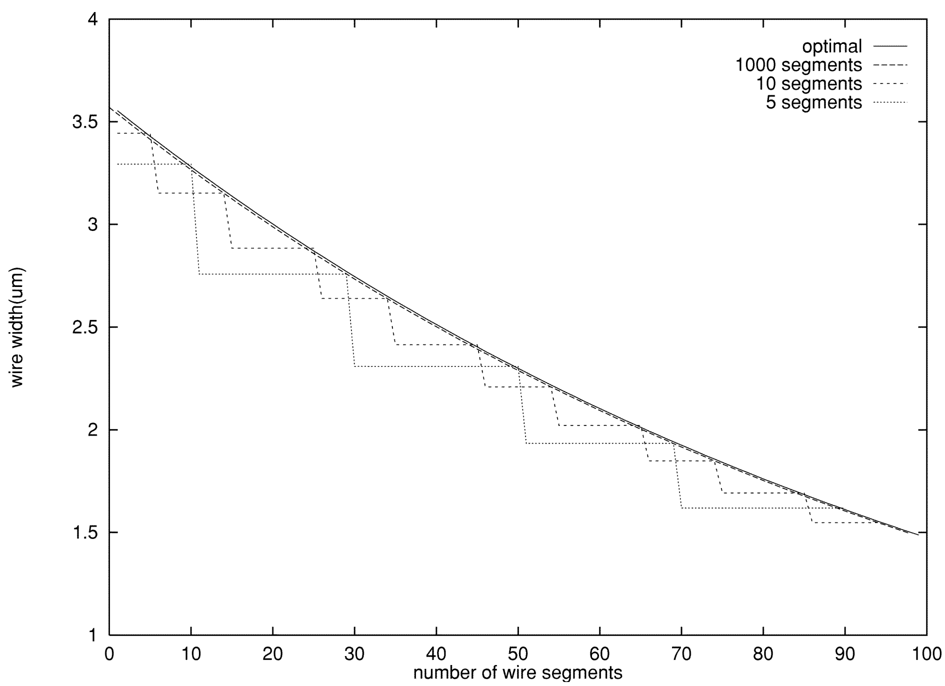

Fig. 9. Approximating nonuniform wire sizing by uniform wire sizing.

V. Experimental Results and Concluding Remarks

We implemented and tested our algorithm using C on a Sun Sparc 5 workstation with 16-MB memory. The parameters used are shown in Table III. The results are given in Table IV. The first column labeled "Precision Requirement" specifies the required accuracy of the wire width values. The second column shows the number of Newton–Raphson iterations. Our results show that even under very strict precision requirements, the number of iterations is at most seven. Thus, in practice, the optimal wire-sizing functions can be computed in $O(1)$ time and hence our method is extremely fast.

We also performed experiments to compare the nonuniform wire-sizing solutions with the uniform ones in which wires are chopped into different number of segments. The results are shown in Fig. 9. Wire widths are plotted as the functions of positions on the wire segments. It shows that the more segments a wire is chopped into, the closer the solution is to our formula. When the wire is chopped into 1000 segments, it can be shown that the corresponding curve and the nonuniform wire-size curve are almost identical.

TABLE V
Runtime and Memory Usage of GWSA-C. [Unit: Runtime (Millisecond), Memory (Kilobytes), Step Width (1 μm)]

# of wire segments	GWSA-C		No of Iters
	Time	Memory	
100	10	32	9
200	20	32	10
1000	80	40	10
2000	170	48	12
10000	1120	108	15
20000	2350	188	16
100000	13520	812	18
1000000	141420	7884	19

Finally, we compare the runtime and memory usage of the optimal wire-sizing function with the GWSA-C on a single wire with 100 to 10^6 segments. We use the Newton–Raphson method [8] to determine b for the optimal wire-sizing function. Then, we substitute b into Equation

(25) to get a. The runtime is only 0.021 35 ms, and the memory usage is extremely low. The runtime and memory usage of GWSA-C is listed in Table V. It is obvious that the optimal wire-sizing function runs much more efficiently than GWSA-C.

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Asymptotic Behavior of Delay 2-D Discrete Logistic Systems

Shu Tang Liu and Guanrong Chen

***Abstract*—Asymptotic behavior of all solutions of the delay two-dimensional logistic system $x_{m+1,n} + ax_{m,n+1} = \mu_{mn}x_{mn}(1 - x_{m-\sigma,n-\tau})$ is investigated. Some sufficient conditions for the stability of this equation are derived. Moreover, some sufficient and necessary conditions for oscillations of all solutions of this equation are obtained.**

***Index Terms*—Delay two-dimensional (2-D) logistic system, linearization, oscillation, stability.**

I. INTRODUCTION

In the engineering literature, particularly in the fields of digital filtering, imaging, and spatial dynamical systems, two-dimensional (2-D) discrete systems have been a focused subject for investigation (see, for example, [1]–[3] and [7]–[13], and the references cited therein). Of especially interesting is the delay 2-D discrete logistic system

$$x_{m+1,\,n} + ax_{m,\,n+1} = \mu_{mn}x_{mn}(1 - x_{m-\sigma,\,n-\tau}) \tag{1.1}$$

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where $\{\mu_{mn}\}$ is a double sequence of positive real constants, $\sigma,\ \tau$ are nonnegative integers, and $a \in (-\infty, \infty)$, $m, n \in N_0 = \{0, 1, 2, \ldots\}$. The stability and oscillation of all solutions of (1.1) are important properties, which, however, have not been carefully studied before. This brief is to introduce a linearization method for analysis of the stability of all solutions of system (1.1). Some sufficient and necessary conditions for oscillations of all its solutions will also be derived.

First, observe that in the particular case where $\mu_{mn} = \mu$ and $\sigma = \tau = 0$, system (1.1) becomes

$$x_{m+1,\,n} + ax_{m,\,n+1} = \mu x_{mn}(1 - x_{mn}) \tag{1.2}$$

and, when $a = 0$ and $n = n_0$, it follows from (1.2) that

$$x_{m+1,\,n_0} = \mu x_{mn_0}(1 - x_{mn_0}) \tag{1.3}$$

which is just the familiar simple case of the one-dimensional (1-D) logistic system. Therefore, (1.1) is quite general. Moreover, (1.1) can be regarded as a discrete analog of the following functional partial differential equation:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \lambda u(x,\, y)\left((1 - u(x-\sigma,\, y-\tau) + \frac{2}{\lambda}\right).$$

In fact, this system is a *convection equation* with a forced term in physics. Therefore, qualitative properties of (1.1) may lead to some useful information for analyzing this companion partial differential system.

Throughout, let

$$\Omega = \{-\sigma,\, -(\sigma+1),\, \ldots,\, -1,\, 0\} \times \{-\tau,\, -(\tau+1),\, \ldots,\, -1,\, 0,\, 1,\, 2,\, \ldots,\, n,\, \ldots\}.$$

In the following, the existence and uniqueness of the solution of system (1.1) are discussed, along the line of the studies in [14, pp. 215–216], and [17, pp. 61–62].

Notice that for a given function $\varphi(i, j)$ defined on Ω, it is easy to construct, by induction, a double sequence $\{x_{ij}\}$ that equals $\varphi(i, j)$ on Ω and satisfies system (1.1) for $i, j = 0, 1, 2, \ldots$. Indeed, one can rewrite (1.1) as

$$x_{m+1,\,n} = \mu_{mn}x_{mn}(1 - x_{m-\sigma,\,n-\tau}) - ax_{m,\,n+1}$$

and then use it with given initial conditions to calculate, successively, $x_{10},\, x_{11},\, x_{20},\, x_{12},\, x_{21},\, x_{30},\, \ldots$.

Since

$$\begin{aligned} x_{10} &= \mu_{00}x_{00}(1 - x_{-\sigma,\,-\tau}) - ax_{01} \\ x_{11} &= \mu_{01}x_{01}(1 - x_{-\sigma,\,1-\tau}) - ax_{02} \\ x_{20} &= \mu_{10}x_{10}(1 - x_{1-\sigma,\,-\tau}) - ax_{11} \\ x_{12} &= \mu_{02}x_{02}(1 - x_{-\sigma,\,2-\tau}) - ax_{03} \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \end{aligned}$$

such a double sequence is unique and is a solution of system (1.1) subject to the *initial condition*

$$x_{ij} = \varphi(i,\, j), \qquad (i,\, j) \in \Omega. \tag{1.4}$$

Let

$$\Omega' = \{(i,\, j) \mid \qquad i,\, j = 0,\, 1,\, 2,\, \ldots\}.$$