

Optimal Wire-Sizing Function with Fringing Capacitance Consideration

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Abstract

In this paper, we consider non-uniform wire-sizing under the Elmore delay model. Given a wire segment of length \mathcal{L} , let $f(x)$ be the width of the wire at position x , $0 \leq x \leq \mathcal{L}$. It was shown in [2, 5] that the optimal wire-sizing function which minimizes delay is an exponential tapering function $f(x) = ae^{-bx}$, where $a > 0$ and $b > 0$ are constants. Unfortunately, [2, 5] did not consider fringing capacitance which is at least comparable in size to area capacitance in deep submicron designs. As a result, exponential tapering is no longer the optimal strategy. In this paper, we show that the optimal wire-sizing function, taking fringing capacitance into consideration, is $f(x) = \frac{-c_f}{2c_0} (\frac{1}{\mathcal{W}(\frac{-c_f}{ae^{-bx}})} + 1)$ where

$\mathcal{W}(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$ is the Lambert's \mathcal{W} function, c_f and c_0 are the respective fringing capacitance and area capacitance of wire per unit square, $a > 0$ and $b > 0$ are constants. The optimal wire-sizing function degenerates into an exponential tapering function as $c_f = 0$, and degenerates into a square-root tapering function ($f(x) = \sqrt{b - ax}$, where $a > 0$ and $b > 0$) as $c_f \rightarrow \infty$. Our experimental results show that the optimal wire-sizing function can significantly reduce the interconnection delay of exponentially tapered wires. In the case where lower and upper bounds on the wire widths are given, the optimal wire-sizing function is a truncated version of the above function. Finally, our optimal wire-sizing function can be iteratively applied to optimally size all the wire segments in a routing tree for objectives such as minimizing weighted sink delay, minimizing maximum sink delay, or minimizing area subject to delay bounds at the sinks.

1 Introduction

As VLSI technology continues to scale down, interconnect delay has become a major concern in deep submicron design. With 70-80% of the system delay comes from the interconnects, it is beneficial to size wires. In this paper, we consider non-uniform wire-sizing under the Elmore delay model [12]. Given a wire segment W of length \mathcal{L} , a source with driver resistance R_d , and a sink with load capacitance C_L . For each

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$x \in [0, \mathcal{L}]$, let $f(x)$ be the width of the wire at position x . Figure 1 shows an example. Let r_0 , c_0 , and c_f be wire resistance per unit square, wire area capacitance per unit square, and fringing capacitance per unit length, respectively. Let D be the Elmore delay from the source to the sink of W . It was

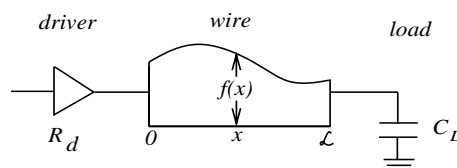


Figure 1: Wire-sizing function.

shown in [2, 5] that the optimal wire-sizing function which minimizes the delay D is an exponential tapering function $f(x) = ae^{-bx}$, where $a > 0$ and $b > 0$ are constants. Unfortunately, [2, 5] did not consider fringing capacitance which is at least comparable in size to area capacitance in deep submicron designs. [11] shows that for a single interconnection line placed on bulk silicon, the capacitance per unit length can be approximated by $C = \epsilon_{ox} [1.15(\frac{W}{H}) + 2.80(\frac{T}{H})^{0.222}]$, where $\epsilon_{ox} = 3.9 * 8.855 * 10^{-14} F/cm$ is the dielectric constant of the insulator such as SiO_2 , W and T are the respective width and thickness of the wire, H is the distance from the wire to the bulk. The first term in the above equation is the area capacitance and the second term is the fringing capacitance. Note that when $T = H = 1\mu m$ and $W = 0.5\mu m$, the fringing capacitance dominates the area capacitance. Clearly, exponential tapering is no longer the optimal strategy.

In this paper, we show that the optimal wire-sizing function, taking fringing capacitance into consideration, is $f(x) = \frac{-c_f}{2c_0} (\frac{1}{\mathcal{W}(\frac{-c_f}{ae^{-bx}})} + 1)$ where $\mathcal{W}(x)$ is the Lambert's \mathcal{W} function, $a > 0$ and $b > 0$ are constants. The Lambert's \mathcal{W} function [7, 8] was first introduced by Euler in 1779 [9] when he studied Lambert's transcendental equation in [10]. $\mathcal{W}(x)$ function is defined as the value of w that satisfies $we^w = x$. Like exponential function, the \mathcal{W} function is differentiable and integrable. For $|x| \leq \frac{1}{e}$, the \mathcal{W} function has the following series expansion

$$\mathcal{W}(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n. \quad (1)$$

The optimal wire-sizing function degenerates into an exponential tapering function as $c_f = 0$, and degenerates into a square-root tapering function ($f(x) = \sqrt{b - ax}$ where $a > 0$ and $b > 0$) as $c_f \rightarrow \infty$. Our experimental results show that the optimal wire-sizing function can significantly reduce the

interconnection delay of exponentially tapered wires. In the case where lower and upper bounds on the wire widths are given, the optimal wire-sizing function is a truncated version of the above function. Finally, our optimal wire-sizing function can be iteratively applied to optimally size all the wire segments in a routing tree for objectives such as minimizing weighted sink delay, minimizing maximum sink delay, or minimizing area subject to delay bounds at the sinks.

Due to space limitation, we omit the proofs of many lemmas and theorems. See [4] for complete proofs.

2 Elmore Delay Model

We use the Elmore delay model [12]. Suppose W is partitioned into n equal-length wire segments, each of length $\Delta x = \frac{\mathcal{L}}{n}$. Let x_i be $i\Delta x$, $1 \leq i \leq n$. The capacitance and resistance of wire segment i can be approximated by $(c_0 f(x_i) + c_f)\Delta x$ and $r_0 \Delta x / f(x_i)$, respectively. Thus the Elmore delay through W can be approximated by

$$D_n = R_d(C_L + \sum_{i=1}^n (c_0 f(x_i) + c_f)\Delta x) + \sum_{i=1}^n \frac{r_0 \Delta x}{f(x_i)} \left(\sum_{j=i}^n (c_0 f(x_j) + c_f)\Delta x + C_L \right).$$

The first term is the delay of the driver, which is given by the driver resistance R_d multiplied by the total capacitance of W and C_L . The second term is the sum of the delay in each wire segment i , which is given by its own resistance $r_0 \Delta x / f(x_i)$ multiplied by its downstream capacitance $\sum_{j=i}^n (c_0 f(x_j) + c_f)\Delta x + C_L$. As $n \rightarrow \infty$, $D_n \rightarrow D$ where

$$D = R_d(C_L + \int_0^{\mathcal{L}} (c_0 f(x) + c_f) dx) + \int_0^{\mathcal{L}} \frac{r_0}{f(x)} \left(\int_x^{\mathcal{L}} (c_0 f(t) + c_f) dt + C_L \right) dx$$

is the Elmore delay through the driver and W .

3 Optimal Wire-Sizing Function

In this section, we determine a wire-sizing function that minimizes D . We show that the optimal wire-sizing function satisfies a differential equation which can be analytically solved.

Theorem 1 *Let f be the optimal wire-sizing function. We have*

$$f^2(x) = \frac{r_0(C_L + c_0 \int_x^{\mathcal{L}} f(t) dt + c_f(\mathcal{L} - x))}{c_0(R_d + r_0 \int_0^x \frac{1}{f(t)} dt)} \quad (2)$$

Note that let $\mathcal{C}(x) = C_L + c_0 \int_x^{\mathcal{L}} f(t) dt + c_f(\mathcal{L} - x)$ be the downstream capacitance at point x and let $\mathcal{R}(x) = R_d + r_0 \int_0^x \frac{1}{f(t)} dt$ be upstream resistance at point x . We can rewrite Equation (2) as follows:

$$f(x) = \sqrt{\frac{r_0 \mathcal{C}(x)}{c_0 \mathcal{R}(x)}}. \quad (3)$$

Since \mathcal{C} is strictly decreasing and \mathcal{R} is strictly increasing, therefore f is strictly decreasing.

By rearranging the terms in (2) and differentiating it with respect to x twice, we get the following theorem.

Theorem 2 *Let $f(x)$ be the optimal wire-sizing function. We have*

$$f''(x)f(x) = f'(x)^2 \left(\frac{2c_0 f(x) - c_f}{2c_0 f(x) + c_f} \right). \quad (4)$$

The following theorem gives the optimal wire-sizing function for the special cases where $c_f = 0$ or $c_f \rightarrow \infty$.

Theorem 3 *If $c_f = 0$, the optimal wire-sizing function is given by an exponential tapering function $f(x) = ae^{-bx}$, where $a > 0$ and $b > 0$. If $c_f \rightarrow \infty$, the optimal wire-sizing function is given by a square root tapering function $f(x) = \sqrt{b - ax}$, where $a > 0$ and $b > 0$.*

For the general case, we can analytically solve the differential equation in (4) and obtain a closed-form solution as shown below.

Theorem 4 *Let f be the optimal wire-sizing function. We have $f(x) = \frac{-c_f}{2c_0} \left(\frac{1}{\mathcal{W}(\frac{-c_f}{ae^{-bx}})} + 1 \right)$, where $a > 0$, $b > 0$, and*

$\mathcal{W}(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$ *is the Lambert's \mathcal{W} function.*

Proof: Let $y = f(x)$ and $P = y'$. We have $y'' = P \frac{dP}{dy}$. We can rewrite Equation (4) as follows:

$$yP \frac{dP}{dy} = P^2 \frac{2c_0 y - c_f}{2c_0 y + c_f}.$$

After separating the variables and integrating, we get

$$P = k_1 \frac{(2c_0 y + c_f)^2}{y}, \text{ where } k_1 \text{ is a constant.}$$

Since $P = \frac{dy}{dx}$, we have

$$\frac{dy}{dx} = k_1 \frac{(2c_0 y + c_f)^2}{y}.$$

By separating the variables and integrating, we get

$$(2c_0 y + c_f) e^{\frac{c_f}{2c_0 y + c_f}} = ae^{-bx}, \text{ where } a > 0 \text{ and } b > 0$$

Let $Y = 2c_0 y + c_f$ and substitute into the above equation, we obtain $Y e^{\frac{c_f}{Y}} = ae^{-bx}$. Equivalently, we have

$$\frac{-c_f}{Y} e^{\frac{-c_f}{Y}} = \frac{-c_f}{ae^{-bx}} \quad (5)$$

Let $w = \frac{-c_f}{Y}$ and $t = \frac{-c_f}{ae^{-bx}}$. We simplify Equation (5) to

$$w e^w = t. \quad (6)$$

By the definition of the Lambert's \mathcal{W} function, we can rewrite Equation (6) as $\mathcal{W}(t) = w$. Thus,

$$\mathcal{W}\left(\frac{-c_f}{ae^{-bx}}\right) = \frac{-c_f}{Y} = \frac{-c_f}{2c_0 y + c_f}.$$

After rearranging the terms, we get

$$y = \frac{-c_f}{2c_0} \left(\frac{1}{\mathcal{W}\left(\frac{-c_f}{ae-bx}\right)} + 1 \right)$$

Finally, We can prove that $|\frac{-c_f}{ae-bx}| \leq \frac{1}{e}$ and hence $\frac{-c_f}{ae-bx}$ is within the radius of convergence of the series expansion of the \mathcal{W} function. See [4] for the proof. \square

We now show how to determine the values of a and b .

Lemma 1 Let $f(t) = \frac{-c_f}{2c_0} \left(\frac{1}{\mathcal{W}\left(\frac{-c_f}{ae-bt}\right)} + 1 \right)$, $\alpha = \mathcal{W}\left(\frac{-c_f}{a}\right)$, $\beta = \mathcal{W}\left(\frac{-c_f}{ae-b\mathcal{L}}\right)$. We have

$$\int_0^{\mathcal{L}} f(t) dt = \frac{-c_f \mathcal{L}}{2c_0} \left(\frac{1}{\alpha} - \frac{1}{\beta} + \ln \frac{\beta}{\alpha} + 1 \right) \quad (7)$$

$$\int_0^{\mathcal{L}} \frac{1}{f(t)} dt = \frac{-2c_0 \mathcal{L}}{c_f} \frac{\beta - \alpha}{\ln \frac{\beta}{\alpha} + \beta - \alpha} \quad (8)$$

Clearly, $f(0)$ and $f(\mathcal{L})$ can be directly written in terms of α and β . Using Equation (2) and Lemma 1, we get alternative expressions for $f(0)$ and $f(\mathcal{L})$ in terms of α and β . By equating the equivalent expressions for $f(0)$ and $f(\mathcal{L})$, we get the following theorem which shows how to determine the constants a and b in the optimal wire-sizing function.

Theorem 5 Let $f(x) = \frac{-c_f}{2c_0} \left(\frac{1}{\mathcal{W}\left(\frac{-c_f}{ae-bx}\right)} + 1 \right)$, where $a > 0$ and $b > 0$, be the optimal wire-sizing function. The constants a and b are the roots of the following equations:

$$\frac{c_f^2 R_d}{4c_0 r_0} \left(1 + \frac{1}{\alpha}\right)^2 = C_L + \frac{-c_f \mathcal{L}}{2} \frac{\frac{1}{\alpha} - \frac{1}{\beta} + \alpha - \beta}{\ln \frac{\beta}{\alpha} + \beta - \alpha} \quad (9)$$

$$\frac{c_f^2}{4c_0 r_0} \left(1 + \frac{1}{\beta}\right)^2 = \frac{C_L}{R_d + \frac{-2c_0 r_0 \mathcal{L}}{c_f} \frac{\beta - \alpha}{\ln \frac{\beta}{\alpha} + \beta - \alpha}}, \quad (10)$$

where $\alpha = \mathcal{W}\left(\frac{-c_f}{a}\right)$ and $\beta = \mathcal{W}\left(\frac{-c_f}{ae-b\mathcal{L}}\right)$.

Remark 1. We apply Newton-Raphson method to Equations (9) and (10) to solve for a and b . A good initial guess can be obtained as follows: First, chop the line into a few segments and apply the GWSA-C wire-sizing algorithm [3] to optimally size the segments (each with uniform width), then extract the initial guess of a and b from the solution. Since Equations (9) and (10) indirectly depend on a and b , the partial derivatives with respect to a and b cannot be obtained by directly differentiating. In order to apply chain rule, we need to compute $\frac{\partial \alpha}{\partial a}$, $\frac{\partial \alpha}{\partial b}$, $\frac{\partial \beta}{\partial a}$, and $\frac{\partial \beta}{\partial b}$. By $\alpha = \mathcal{W}\left(\frac{-c_f}{a}\right)$ and $\beta = \mathcal{W}\left(\frac{-c_f}{ae-b\mathcal{L}}\right)$, we know

$$\alpha e^\alpha = \frac{-c_f}{a}, \quad \beta e^\beta = \frac{-c_f}{ae-b\mathcal{L}}. \quad (11)$$

By differentiating both sides of each of the equations in (11) with respect to a and b , we get

$$\frac{\partial \alpha}{\partial a} = \frac{c_f}{(1+\alpha)a^2 e^\alpha}, \quad \frac{\partial \alpha}{\partial b} = 0, \\ \frac{\partial \beta}{\partial a} = \frac{c_f}{(1+\beta)a^2 e^{\beta-b\mathcal{L}}}, \quad \frac{\partial \beta}{\partial b} = \frac{bc_f}{(1+\beta)ae^{\beta-b\mathcal{L}}}.$$

Remark 2. Figure 2 shows a typical shape of the wire-sizing function. In general, the optimal wire-sizing function can be roughly divided into three regions. In region I, since the wire has larger width, area capacitance dominates fringing capacitance. Hence the shape of the function is similar to that of exponential tapering. In region II, since the wire is of medium width, area capacitance and fringing capacitance play competitive roles. Hence the shape of the function is similar to a combination of the exponential tapering and square root tapering which looks like a straight line. In region III, since the width of the wire is smaller, fringing capacitance dominates area capacitance, hence the shape of the wire-sizing function is similar to that of square root tapering.

Remark 3. Recently, [6] independently determined the optimal wire-sizing function using calculus of variation. The optimal wire-sizing function was expressed as a power series whose coefficients had to be computed one at a time by symbolically or numerically integrating. In contrast, we are giving a closed-form solution to the wire-sizing problem.

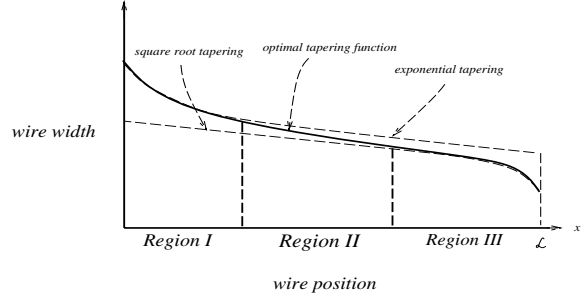


Figure 2: Optimal wire-sizing function.

4 Constrained Wire-Sizing

In constrained wire-sizing, we are given $0 < L < U < \infty$, and require that $L \leq f(x) \leq U$, $0 \leq x \leq \mathcal{L}$. It is clear that if the wire-sizing function $f(x)$ obtained for the unconstrained case lies within bounds L and U , then $f(x)$ is also optimal for constrained wire sizing. On the other hand, if for some x , $f(x)$ is not in $[L, U]$, a simple approach is to round $f(x)$ to either L or U ; i.e. the new function is obtained by a direct truncation of $f(x)$ by $y = L$ and $y = U$. Unfortunately, an argument similar to the one used in [2] shows that the resulting function is not optimal. However, as in [2], it can be shown that the optimal constrained wire-sizing function is continuous and decreasing. As a result, the optimal wire-sizing function $f(x)$ consists of (at most) three parts. The first part is $f(x) = U$, the middle part is a decreasing function, and the last part is $f(x) = L$. The three parts of $f(x)$ partition W into three wire segments, A , B , and C , where A has width U , C has width L , and B is defined by the middle part of $f(x)$. It is easy to see that the middle part of $f(x)$ must be of the form $f(x) = \frac{-c_f}{2c_0} \left(\frac{1}{\mathcal{W}\left(\frac{-c_f}{ae-b(x-t_1)}\right)} + 1 \right)$ for some $a > 0$ and $b > 0$. To see this, we can consider the wire segment A to be a part of the driver and its resistance to be a part of R_d . Similarly, the wire segment C can be considered as a part of the load and its capacitance as a part of C_L . According to Equation (9), we can recalculate a and b using the new values of R_d and

C_L , as long as we know the length of the wire segments A and B . Let l_1 , l_2 , and l_3 be the length of wire segments A , B , and C , respectively. The optimal wire-sizing function is given as follows:

$$f(x) = \begin{cases} U & \text{if } 0 \leq x \leq l_1 \\ \frac{-c_f}{2c_0} \left(\frac{1}{W\left(\frac{-c_f}{ae^{-b(x-l_1)}}\right)} + 1 \right) & \text{if } l_1 \leq x \leq l_1 + l_2 \\ L & \text{if } l_1 + l_2 \leq x \leq \mathcal{L} \end{cases}$$

where a , b , l_1 , l_2 , and l_3 are constants whose values depend on the input parameters c_0 , c_f , r_0 and \mathcal{L} . See [4] for the details in determining these constants.

5 Application to Routing Trees

Recently, [1] applied the wire-sizing formula in [2] to size routing trees under the Elmore delay model. As in [2], [1] did not consider fringing capacitance, i.e. $c_f = 0$. Three minimization objectives were studied: 1) total weighted sink-delays; 2) total area subject to sink-delay bounds; and 3) maximum sink-delay. [1] presented an algorithm NWSA-wd for minimizing total weighted sink-delays based on iteratively applying the wire-sizing function in [2] to size one wire segment at a time. Whenever the wire-sizing function in [2] is used to size a wire segment in the tree, R_d is set to be the total weighted upstream resistance, including the driver resistance, and C_L is set to be the total weighted downstream capacitance, including the load capacitances of the sinks in the subtree. It was shown in [1] that NWSA-wd always converges to an optimal wire-sizing solution. Based on NWSA-wd and the Lagrangian relaxation technique, [1] presented two other algorithms NWSA-db and NWSA-md which can optimally solve the other two minimization objectives. In order to take fringing capacitance into consideration, we can simply use the optimal wire-sizing function presented in this paper instead of the one in [2] in NWSA-wd, NWSA-bd, and NWSA-md. The three algorithms NWSA-wd, NWSA-bd, and NWSA-md, with the above modification, would again give globally optimal wire-sizing solutions for routing trees (with respect to the above three minimization objectives).

6 Experimental Results

We implemented our algorithm in C on a Sun Sparc 5 workstation. The parameters in our experiment are as follows: $r_0 = 0.03\Omega/\mu m$, $c_0 = 0.2fF/\mu^2 m$, $c_f = 0fF/\mu m - 0.5$, $R_d = 1\Omega$, and $C_L = 20fF$. In Figure 3, we show the optimal wire-sizing functions for different values of c_f . It shows that when $c_f = 0$, the optimal wire-sizing function degenerates to exponential tapering. It also shows as c_f increases, the optimal function gradually changes from exponential tapering to square root tapering. The delay comparisons with respect to different values of c_f and different ways of tapering (minimum width sizing, exponential tapering and optimal tapering) are shown in Table 1. It shows that the optimal wire-sizing function can even reduce half of the delay of the exponential wire-sizing function when $c_f = c_0$. The runtime of our program is always within 0.01 cpu seconds.

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c_f/c_0	Delay (ps)		
	Min-Width	Exponential	Optimal
0.0	43	28	28
0.5	335	123	80
1.0	627	214	117
1.5	920	309	150
2.0	211	400	182

Table 1: The delay comparisons between minimum width sizing, exponential tapering, and the optimal wire-sizing function with $3000\mu m$ wires .

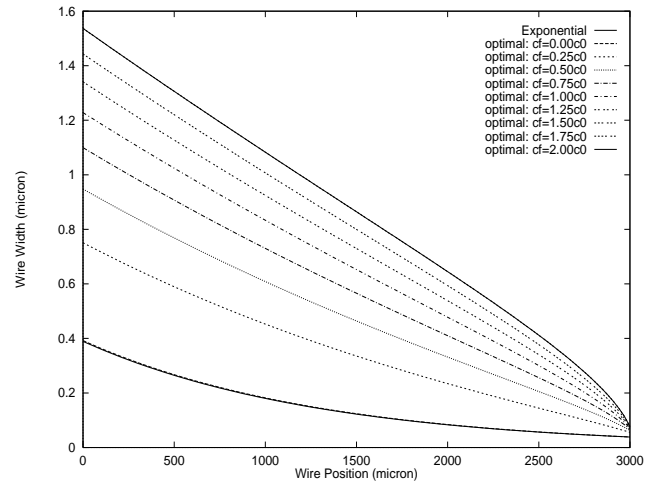


Figure 3: The optimal wire-sizing function with respect to different fringing capacitance values.

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