Optimal Wire-Sizing Formula Under the Elmore Delay Model *

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Abstract

In this paper, we consider non-uniform wire-sizing. Given a wire segment of length $\mathcal{L}$, let $f(x)$ be the width of the wire at position x, $0 \le x \le \mathcal{L}$. We show that the optimal wire-sizing function that minimizes the Elmore delay through the wire is $f(x) = ae^{-bx}$, where $a > 0$ and $b > 0$ are constants that can be computed in $O(1)$ time. In the case where lower bound ($L > 0$) and upper bound ($U > 0$) on the wire widths are given, we show that the optimal wire-sizing function $f(x)$ is a truncated version of ae^{-bx} that can also be determined in $O(1)$ time. Our wire-sizing formula can be iteratively applied to optimally size the wire segments in a routing tree.

1 Introduction

As VLSI technology continues to scale down, interconnect delay has become the dominant factor in deep submicron designs. As a result, wire-sizing plays an important role in achieving desirable circuit performance. Recently, many wire-sizing algorithms have been reported in the literature [1, 2, 4, 5, 7]. All these algorithms size each wire segment uniformly, i.e., identical width at every position on the wire. In order to achieve non-uniform wire-sizing, existing algorithms have to chop wire segments into large number of small segments. Consequently, the number of variables in the optimization problem is increased substantially and thus results in long runtime and large storage.

In this paper, we consider non-uniform wire-sizing. Given a wire segment W of length $\mathcal{L}$, a source with driver resistance R_d, and a sink with load capacitance C_L. For each $x \in [0, L]$, let $f(x)$ be the wire width of W at position x. Figure 1 shows an example. Let r_0 and c_0 be the respective wire resistance and wire capacitance per unit square. Let D be the Elmore delay from the source to the sink of W. We show that the optimal wire-sizing function $f(x)$ that minimizes D satisfies an ordinary differential equation which can be analytically solved. We have $f(x) = ae^{-bx}$, where $a > 0$ and $b > 0$ are constants that can be computed in $O(1)$ time. These constants depend on $R_d, C_L, \mathcal{L}, r_0$, and c_0. In the case where lower bound ($L > 0$) and upper bound ($U > 0$) on the wire widths are given, i.e. $L \le f(x) \le U$, $0 \le x \le \mathcal{L}$, we show that the optimal wire-sizing function $f(x)$ is a truncated version of ae^{-bx} that can also be determined in $O(1)$ time. Our wire-sizing formula can be iteratively applied to optimally size the wire segments in a routing tree.

2 Optimal Wire-Sizing Function

We use the Elmore delay model [3]. Suppose W is partitioned into n equal-length wire segments, each of length $\triangle x$

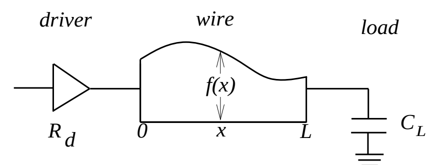

Figure 1: Non-uniform wire-sizing.

$= \frac{\mathcal{L}}{n}$. Let x_i be $i\triangle x$, $1 \le i \le n$. The capacitance and resistance of wire segment i can be approximated by $c_0 \triangle x f(x_i)$ and $r_0 \triangle x / f(x_i)$, respectively. Thus the Elmore delay through W can be approximated by

$$D_n = R_d(C_L + \sum_{i=1}^{n} c_0 f(x_i)\triangle x) + \sum_{i=1}^{n} \frac{r_0 \triangle x}{f(x_i)} (\sum_{j=i}^{n} c_0 f(x_j)\triangle x + C_L).$$

The first term is the delay in the driver, which is given by the driver resistance R_d multiplied by the total capacitance of W and C_L. The second term is the sum of the delay in each wire segment i, which is given by its own resistance $r_0 \triangle x / f(x_i)$ multiplied by its downstream capacitance $\sum_{j=i}^{n} c_0 f(x_j) \triangle x + C_L$. As $n \to \infty$, $D_n \to D$ where

$$D = R_d(C_L + \int_0^{\mathcal{L}} c_0 f(x) dx) + \int_0^{\mathcal{L}} \frac{r_0}{f(x)} \ (\int_x^{\mathcal{L}} c_0 f(t)\ dt + C_L)\ dx$$

is the Elmore delay through W.

In this section, we derive closed-form formula for the optimal wire-sizing function $f(x)$. We consider two cases: *unconstrained* and *constrained wire-sizing*. In unconstrained wire-sizing, there is no bound on the value of $f(x)$; i.e. we determine $f : [0, \mathcal{L}] \to (0, \infty)$ that minimizes D. In constrained wire-sizing, we are given $L > 0$ and $U < \infty$, and require that $L \le f(x) \le U$, $0 \le x \le \mathcal{L}$; i.e., we determine $f : [0, \mathcal{L}] \to [L, U]$ that minimizes D.

2.1 Unconstrained Wire-Sizing

We now derive the optimal unconstrained wire-sizing function.

Theorem 1 *Let $f(x)$ be an optimal wire-sizing function. We have*

$$f^2(x) \quad = \quad \frac{r_0(C_L + c_0 \int_x^{\mathcal{L}} f(t)dt)}{c_0(R_d + r_0 \int_0^x \frac{1}{f(t)} dt)}. \tag{1}$$

Note that $C_L + c_0 \int_x^{\mathcal{L}} f(t)dt$ is equal to the downstream capacitance at point x (denoted by Γ_x) and $R_d + r_0 \int_0^x \frac{1}{f(t)} dt$ is

*This work was partially supported by the Texas Advanced Research Program under Grant No. 003658459.

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equal to the upstream resistance at point x (denoted by Φ_x). Hence we can rewrite Equation (1) as follows:

$$f(x) = \sqrt{\frac{r_0 \Gamma_x}{c_0 \Phi_x}}. \tag{2}$$

By rearranging the terms in (1) and differentiating it with respect to x twice, we get the following theorem.

Theorem 2 *Let $f(x)$ be an optimal wire-sizing function. We have*

$$f''(x) f(x) \quad = \quad f'(x)^2. \tag{3}$$

The following theorem shows that the differential equation in (3) has a closed-form solution.

Theorem 3 *Let $f(x) = ae^{-bx}$, where $a = \frac{r_0}{bR_d}$ and*

$$b\sqrt{\frac{R_d C_L}{r_0 c_0}} - e^{\frac{-b\mathcal{L}}{2}} = 0. \tag{4}$$

We have, $f(x)$ is an optimal wire-sizing function.

Note that the function $g(z) = z\sqrt{\frac{R_d C_L}{r_0 c_0}} - e^{\frac{-z\mathcal{L}}{2}}$ is a strictly increasing function in z, $g(0) < 0$, and $\lim_{z\to\infty} g(z) > 0$. Thus $g(z)$ has a unique root $b > 0$. We can use Newton-Raphson method to determine b and, in practice, five to seven iterations are sufficient. Since $a = \frac{r_0}{R_d b}$ and $b > 0$, $a > 0$. Figure 2 shows the exponentially decreasing nature of the optimal wire-sizing function.

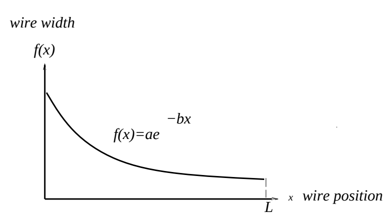

Figure 2: Optimal unconstrained wire-sizing.

2.2 Constrained Wire-Sizing

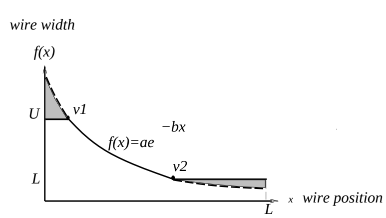

Figure 3: Direct truncation according to U and L is not optimal.

We now consider constrained wire-sizing. It is clear that if the wire-sizing function $f(x)$ obtained for the unconstrained case lies within bounds L and U, then $f(x)$ is also optimal for constrained wire sizing. On the other hand, if for some x, $f(x)$ is not in $[L, U]$, a simple approach is to round $f(x)$ to either L or U; i.e. the new function is obtained by a direct truncation of $f(x)$ by $y = L$ and $y = U$. (See Figure 3.) Unfortunately, the resulting function is not optimal. The reason is that, when we consider the point v_1, which is the intersection of the curves $f(x) = ae^{-bx}$ and $y = U$, from Equation (2) v_1 must satisfy

$$f(v_1) = \sqrt{\frac{r_0 \Gamma_{v_1}}{c_0 \Phi_{v_1}}}, \tag{5}$$

for v_1 to be on the optimal curve. However, from Figure 3, it is clear that v_1 does not satisfy Equation (5), because both of its upstream resistance and downstream capacitance should be recalculated according to the new function, in which the two values associated with v_1 are reduced because of the truncation. Thus this simple approach is not optimal.

Note that the optimal unconstrained wire-sizing function is a decreasing function. We can show that the optimal constrained wire-sizing function must also be decreasing.

Theorem 4 *Let $f(x)$ be an optimal constrained wire-sizing function. We have, $f(x)$ is decreasing on $[0, L]$.*

According to Theorem 4, the optimal wire-sizing function $f(x)$, similar to the one shown in Figure 3, consists of (at most) three parts. The first part is $f(x) = U$, the middle part is a decreasing function, and the last part is $f(x) = L$. The three parts of $f(x)$ partition W into three wire segments, A, B, and C, where A has width U, C has width L, and B is defined by the middle part of $f(x)$. It is easy to see that the middle part of $f(x)$ must be of the form $f(x) = ae^{-bx}$ for some $a > 0$ and $b > 0$. To see this, we can consider the wire segment A to be a part of the driver and its resistance to be a part of R_d. Similarly, the wire segment C can be considered as a part of the load and its capacitance as a part of C_L. According to Equation (4), we can recalculate a and b using the new values of R_d and C_L, as long as we know the length of the wire segments A and B.

As mentioned before, not all three parts of $f(x)$ needed to be present. In fact, an optimal constrained wire-sizing function $f(x)$ can be of any one of the six types of functions (type-A, type-B, type-C, type-AB, type-BC, and type-ABC) as shown in Figure 4. Note that the six function types clearly are named after the wire-segment types which are presented in W. For example, in a type-AB function, W consists only of wire segments A and B. As shown in Figure 4, l_1, l_2, and l_3 are the length of wire segments A, B, and C, respectively.

We now define six wire-sizing functions f_A, f_B, f_C, f_{AB}, f_{BC}, f_{ABC} as follows: All six functions are of the form

$$f(x) = \begin{cases} U & , 0 \le x \le l_1, \\ ae^{-bx} & , l_1 \le x \le l_1 + l_2, \\ L & , l_1 + l_2 \le x \le l_1 + l_2 + l_3 = \mathcal{L}, \end{cases}$$

where the parameters a, b, l_1, l_2, and l_3 for the six functions are given in Table 1. Typically, the names of the functions correspond to their types, i.e., f_A is of type-A, f_B is of type-B, and so on, but it is not always true. For example, it is possible that after we compute the parameters for f_{AB} we get $l_1 \ge \mathcal{L}$ and hence it is of type-A; it is also possible that f_{AB} degenerates into a type-B function. In this case, we say that f_{AB} is of the *wrong* type. We also note that sometimes the functions may be *illegal* in the sense that they violate the

	l_1	l_2	l_3	a	b
f_A	$\mathcal{L}$	0	0	U	0
f_B	0	$\mathcal{L}$	0	$\Psi_1(a) = 0$	$\frac{r_0}{aR_d}$
f_C	0	0	$\mathcal{L}$	L	0
f_{AB}	$\Psi_2(l_1) = 0$	$\mathcal{L} - l_1$	0	U	$\frac{r_0}{R_dU + r_0l_1}$
f_{BC}	0	$\mathcal{L} - l_3$	$\Psi_3(l_3) = 0$	$\frac{r_0(C_L + c_0Ll_3)}{R_dc_0L}$	$\frac{c_0L}{C_L + c_0Ll_3}$
f_{ABC}	$\frac{\frac{C_L}{c_0L} + \mathcal{L} - (1 + ln\frac{U}{L})\frac{UR_d}{r_0}}{2 + ln\frac{U}{L}}$	$\frac{ln\frac{U}{L}(\frac{C_L}{c_0L} + \mathcal{L} + \frac{UR_d}{r_0})}{2 + ln\frac{U}{L}}$	$\mathcal{L} - l_2 - l_1$	U	$\frac{r_0}{R_dU + r_0l_1}$
$\Psi_1(a) = a^2 - \frac{R_0C_L}{c_0R_d}e^{\frac{r_0\mathcal{L}}{aR_d}}$; $\Psi_2(l_1) = \frac{r_0\mathcal{L} + R_dU}{R_dU + r_0l_1} - ln\frac{c_0U(R_dU + r_0l_1)}{r_0C_L} - 1$; $\Psi_3(l_3) = \frac{C_L + c_0L\mathcal{L}}{C_L + c_0Ll_3} - ln\frac{r_0(C_L + c_0Ll_3)}{c_0R_dL^2} - 1.$					

Table 1: Definitions of the wire-sizing functions f_A, f_B, f_C, f_{AB}, f_{BC}, and f_{ABC}.

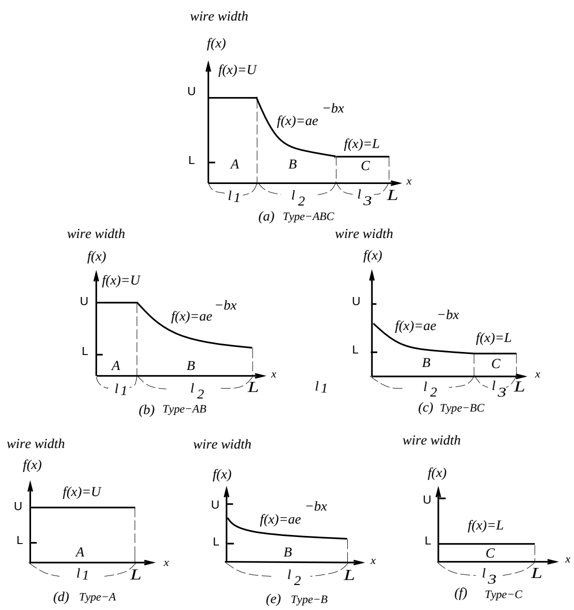

Figure 4: Six types of optimal wire-sizing functions.

wire-width constraints. Nevertheless, we can show that these six functions are candidates for an optimal constrained wire-sizing function $f(x)$. In fact, if we eliminate the functions that are either illegal or of the wrong type, an optimal wire-sizing function can be chosen as the best one (in terms of delay) among the remaining ones. We have the following theorem.

Theorem 5 *Let $G \subseteq F = \{f_A, f_B, f_C, f_{AB}, f_{BC}, f_{ABC}\}$ be the set of functions that are either illegal or of the wrong type. Let $f \in F - G$ be a function which has minimum delay. We have, f is an optimal constrained wire-sizing function.*

The above method always requires the computation of all six functions in F. With the help of additional analysis, we can speed up the procedure. Table 2 shows a set of six *feasibility conditions* $\{\varphi_A, \varphi_B, \varphi_C, \varphi_{AB}, \varphi_{BC}, \varphi_{ABC}\}$ on $\mathcal{L}$. Let $\Gamma = \{A, B, C, AB, CB, ABC\}$.

Lemma 1 *The six feasibility conditions $\{\varphi_A, \varphi_B, \varphi_C, \varphi_{AB}, \varphi_{BC}, \varphi_{ABC}\}$ cover all possible $\mathcal{L} > 0$. Moreover, if $\mathcal{L}$ satisfies φ_z, where $z \in \Gamma$, then f_z is legal and is of type-z.*

Theorem 6 *Let $H = \{f_z | z \in \Gamma$ and $\mathcal{L}$ satisfies $\varphi_z\}$. Let $f \in H$ be a function which has minimum delay. We have, f is an optimal constrained wire-sizing function.*

According to Theorem 6, we only need to check the six feasibility conditions. Only the functions in H needed to be computed. In general, $|H| < 6$; in fact, we have never encountered any case where $|H| > 1$.

φ_A	$\mathcal{L} \le \frac{C_L}{c_0U} - \frac{R_dU}{r_0}$
φ_B	$\mathcal{L} \le min\{\frac{UR_d}{r_0}ln\frac{c_0R_dU^2}{r_0C_L}, \frac{C_L}{r_0L}ln\frac{r_0C_L}{c_0R_dL^2}\}$
φ_C	$\mathcal{L} \le \frac{R_dL}{r_0} - \frac{C_L}{c_0L}$
φ_{AB}	$\mathcal{L} > max\{\frac{UR_d}{r_0}ln\frac{c_0R_dU^2}{r_0C_L}, \frac{C_L}{c_0U} - \frac{R_dU}{r_0}\}$ and $\mathcal{L} \le (1 + ln\frac{U}{L})\frac{C_L}{c_0L} - \frac{R_dU}{r_0}$
φ_{BC}	$\mathcal{L} > max\{\frac{R_dL}{r_0} - \frac{C_L}{c_0L}, \frac{C_L}{r_0L}ln\frac{r_0C_L}{c_0R_dL^2}\}$ and $\mathcal{L} \le (1 + ln\frac{U}{L})\frac{R_dU}{r_0} - \frac{C_L}{c_0L}$
φ_{ABC}	$\mathcal{L} > max\{(1 + ln\frac{U}{L})\frac{R_dU}{r_0} - \frac{C_L}{c_0L}, (1 + ln\frac{U}{L})\frac{C_L}{c_0L} - \frac{R_dU}{r_0}\}$

Table 2: Feasibility Conditions.

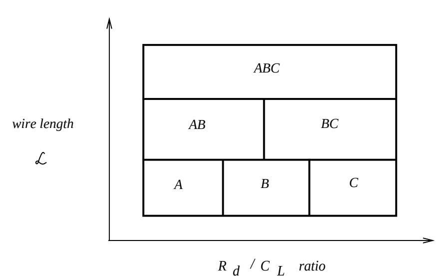

Figure 5: Relationships among the six types of functions with respect to $\mathcal{L}$ and $\frac{R_d}{C_L}$.

We also have the following interesting observations. In Figure 5, we show the relationships among the six types of optimal wire-sizing functions with respect to the three parameters:

wire length $\mathcal{L}$, driver resistance R_d, and load capacitance C_L. The horizontal axis represents the ratio of the driver resistance to the load capacitance. The vertical axis represents the wire length $\mathcal{L}$. Suppose we keep $\frac{R_d}{C_L}$ fixed and varies $\mathcal{L}$. When $\mathcal{L}$ is small, optimal wire-sizing functions tend to be of type-A, type-B, or type-C. As we increase the $\mathcal{L}$, wire-sizing function types will change to type-AB or type-BC when $\mathcal{L}$ is of moderate size and will be of type-ABC when $\mathcal{L}$ is large. Suppose we keep $\mathcal{L}$ fixed and varies $\frac{R_d}{C_L}$. When $\mathcal{L}$ is small, as we increase $\frac{R_d}{C_L}$, optimal wire-sizing function will change from type-A to type-B and then to type-C. When $\mathcal{L}$ is of moderate size, optimal wire-sizing function will change from type-AB to type-BC as we increase $\frac{R_d}{C_L}$. Roughly speaking, the larger the ratio $\frac{R_d}{C_L}$, the smaller the wire sizes. When the wire length $\mathcal{L}$ is very large, optimal wire-sizing function is most likely to be of type-ABC.

3 Application to Routing Trees

Our wire-sizing formula can be applied to size a general routing tree. Recently, [2] presents a wire-sizing algorithm GWSA-C for sizing the wire segments in routing trees. Each segment in the tree is sized uniformly, i.e. uniform wire width per segment. Basically, GWSA-C is an iterative algorithm with guaranteed convergence to a global optimal solution. In each iteration of GWSA-C, the wire segments are examined one at a time; each time a wire segment is uniformly re-sized optimally while keeping the widths of the other segments fixed. We can incorporate our wire-sizing formula into GWSA-C to size each wire segment non-uniformly. When we apply our wire-sizing formula to size a wire segment in a tree, R_d should be set to be the total upstream resistance including the driving resistance, and the C_L should be set to be the total downstream capacitance, including the load capacitances of the sinks in the subtree. It can be shown that this modified algorithm always converges to a global optimal solution.

4 Experimental Results

We implemented and tested our algorithm in C on a Sun Sparc 5 workstation with 16 MB memory. The parameters used are shown in Table 3. The results are given in Table 4. The first column labeled "Precision Requirement" specifies the required accuracy of the wire width values. The second column shows the number of Newton-Raphson iterations. Since the wire-sizing formula can be derived in $O(1)$ time, we are interested in knowing the cost of the numerical method adopted to obtain some parameter values. It shows that even under very strict precision requirement the number of iterations is at most 7. Our method is extremely fast.

We also performed experiments to compare the non-uniform wire-sizing solutions with the uniform ones in which wires are chopped into different number of segments. The results are drawn in Figure 6. Wire widths are plotted as the functions of positions on the wire segments. It shows that the more segments a wire is chopped into, the closer the solution is to our formula. When the wire is chopped into 1000 segments, it can be shown that the corresponding curve and the non-uniform wire-size curve are almost identical.

Unit Resistance:	$0.008\ \ \Omega/\mu m$
Unit Capacitance:	$6E-17\ \ F/\mu m$
Minimum Wire Width:	$1\ \ \mu m$
Maximum Wire Width:	$3.5\ \ \mu m$
Driver Resistance:	$25\ \ \Omega$
Load Capacitance:	$1E-12\ \ F$

Table 3: RC Parameters

Precision Requirement (μm)	# of iterations
0.1	5
0.01	5
0.001	5
0.0001	6
0.00001	6
0.000001	7

Table 4: The number of Newton-Raphson iterations.

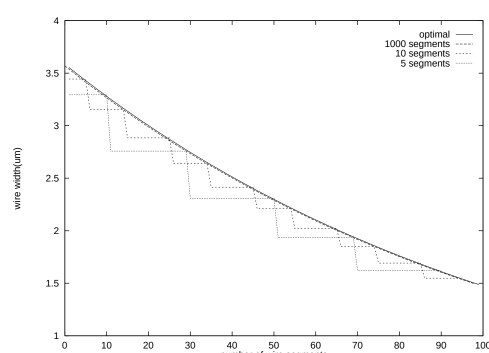

Figure 6: Approximating non-uniform wire-sizing by uniform wire-sizing.

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