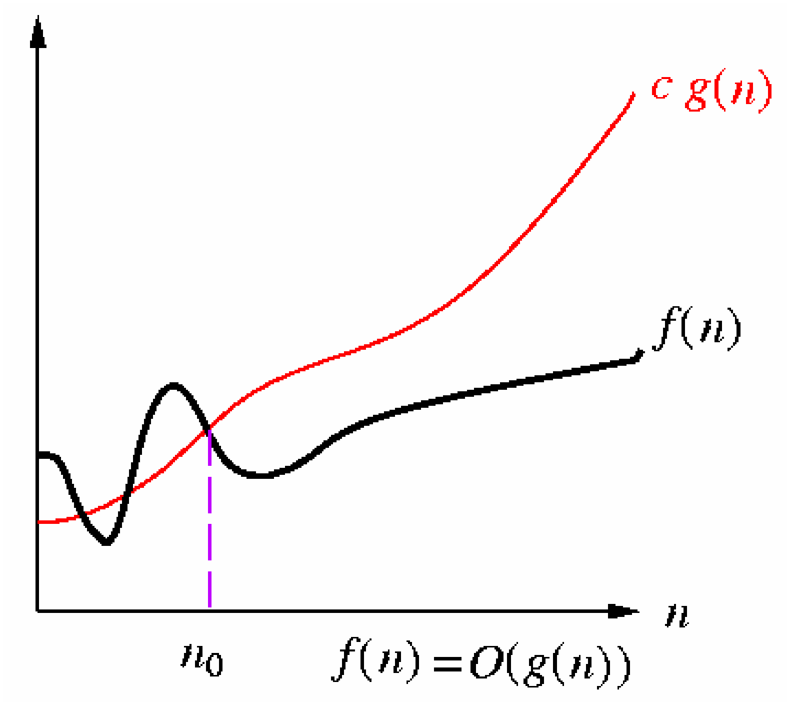
Unit 1A: Computational Complexity

- Course contents:
  - Computational complexity
  - NP-completeness
  - Algorithmic Paradigms
- Readings
  - Chapters 3, 4, and 5

Time	Big-Oh	$n = 10$	$n = 100$	$n = 10^3$	$n = 10^6$
500	$O(1)$	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec
$3n$	$O(n)$	3×10^{-8} sec	3×10^{-7} sec	3×10^{-6} sec	0.003 sec
$n \log n$	$O(n \log n)$	3×10^{-8} sec	2×10^{-7} sec	3×10^{-6} sec	0.006 sec
n^2	$O(n^2)$	1×10^{-7} sec	1×10^{-5} sec	0.001 sec	16.7 min
n^3	$O(n^3)$	1×10^{-6} sec	0.001 sec	1 sec	3×10^5 cent.
2^n	$O(2^n)$	1×10^{-6} sec	3×10^{17} cent.	∞	∞
$n!$	$O(n!)$	0.003 sec	∞	∞	∞

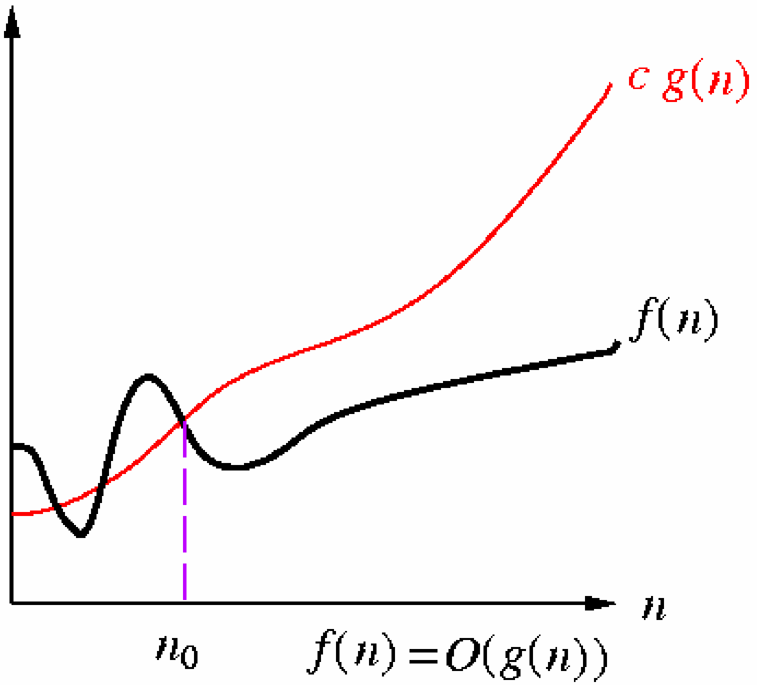
O: Upper Bounding Function

- **Def:** $f(n) = O(g(n))$ if $\exists\ c > 0$ and $n_0 > 0$ such that $0 \le \boldsymbol{f(n)} \le \boldsymbol{cg(n)}$ for all $n \ge n_0$.
  - Examples: $2n^2 + 3n = O(n^2)$, $2n^2 = O(n^3)$, $3n \lg n = O(n^2)$
- Intuition: $f(n)$ "$\le$" $g(n)$ when we ignore constant multiples and small values of n.



Big-*O* Notation

- How to show *O* (Big-Oh) relationships?
  - $f(n) = O(g(n))$ iff $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$ for some $\mathbf{c \geq 0}$.
- "An algorithm has worst-case running time $O(f(n))$": there is a constant c s.t. for every n big enough, **every execution** on an input of size n takes **at most** $cf(n)$ time.



Computational Complexity

- Computational complexity: an abstract measure of the time and space necessary to execute an algorithm as function of its "input size".
- Input size examples:
  - sort n words of bounded length $\Rightarrow n$
  - the input is the integer $n \Rightarrow \lg n$
  - the input is the graph $G(V, E) \Rightarrow |V|$ and $|E|$
- Time complexity is expressed in *elementary computational steps* (e.g., an addition, multiplication, pointer indirection).
- Space Complexity is expressed in *memory locations* (e.g. bits, bytes, words).

Asymptotic Functions

- Polynomial-time complexity: $O(n^k)$, where n is the **input size** and k is a constant.
- Example polynomial functions:
  - 999: constant
  - $\lg n$: logarithmic
  - $\sqrt{n}$: sublinear
  - n: linear
  - $n \lg n$: loglinear
  - n^2: quadratic
  - n^3: cubic
- Example non-polynomial functions
  - 2^n, 3^n: exponential
  - $n!$: factorial

Running-time Comparison

- Assume 1000 MIPS (Yr: 200x), 1 instruction /operation

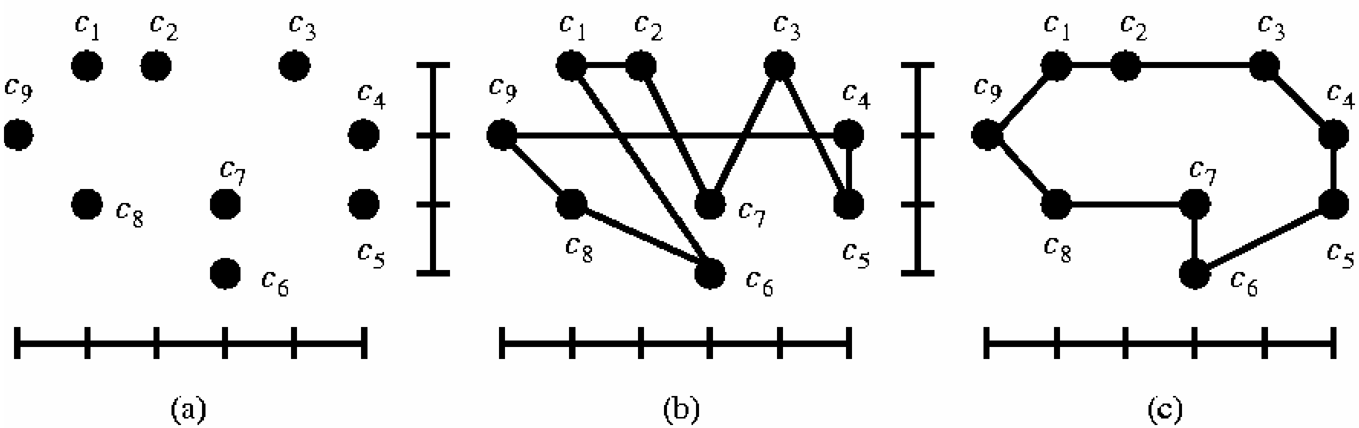
Time	Big-Oh	$n = 10$	$n = 100$	$n = 10^3$	$n = 10^6$
500	$O(1)$	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec
$3n$	$O(n)$	3×10^{-8} sec	3×10^{-7} sec	3×10^{-6} sec	0.003 sec
$n \log n$	$O(n \log n)$	3×10^{-8} sec	2×10^{-7} sec	3×10^{-6} sec	0.006 sec
n^2	$O(n^2)$	1×10^{-7} sec	1×10^{-5} sec	0.001 sec	16.7 min
n^3	$O(n^3)$	1×10^{-6} sec	0.001 sec	1 sec	3×10^5 cent.
2^n	$O(2^n)$	1×10^{-6} sec	3×10^{17} cent.	∞	∞
$n!$	$O(n!)$	0.003 sec	∞	∞	∞

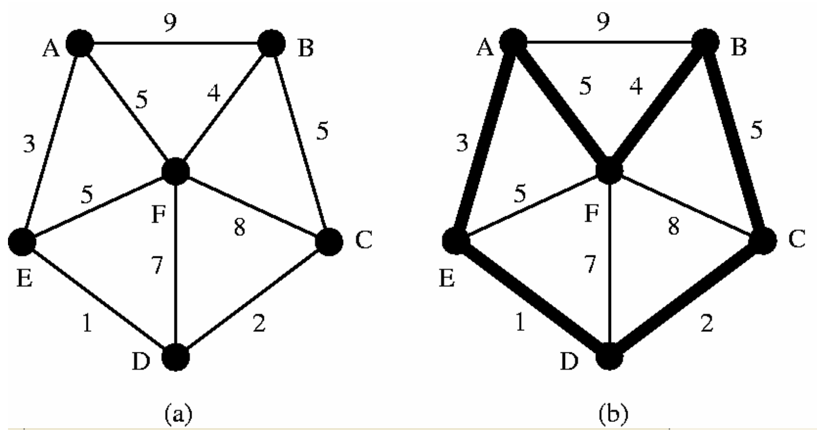
Optimization Problems

- **Problem:** a general class, e.g., "the shortest-path problem for directed acyclic graphs."
- **Instance:** a specific case of a problem, e.g., "the shortest-path problem in a specific graph, between two given vertices."
- **Optimization problems:** those finding a legal configuration such that its cost is minimum (or maximum).
  - MST: Given a graph $G=(V, E)$, find the cost of a minimum spanning tree of G.
- An instance $I = (F, c)$ where
  - F is the set of *feasible solutions*, and
  - c is a *cost function*, assigning a cost value to each feasible solution $c : F \rightarrow R$
  - The solution of the optimization problem is the feasible solution with optimal (minimal/maximal) cost
- c.f., **Optimal** solutions/costs, optimal (**exact**) algorithms (Attn: optimal $\neq$ exact in the theoretic computer science community).

The Traveling Salesman Problem (TSP)

- TSP: Given a set of cities and that distance between each pair of cities, find the distance of a "minimum tour" starts and ends at a given city and visits every city exactly once.



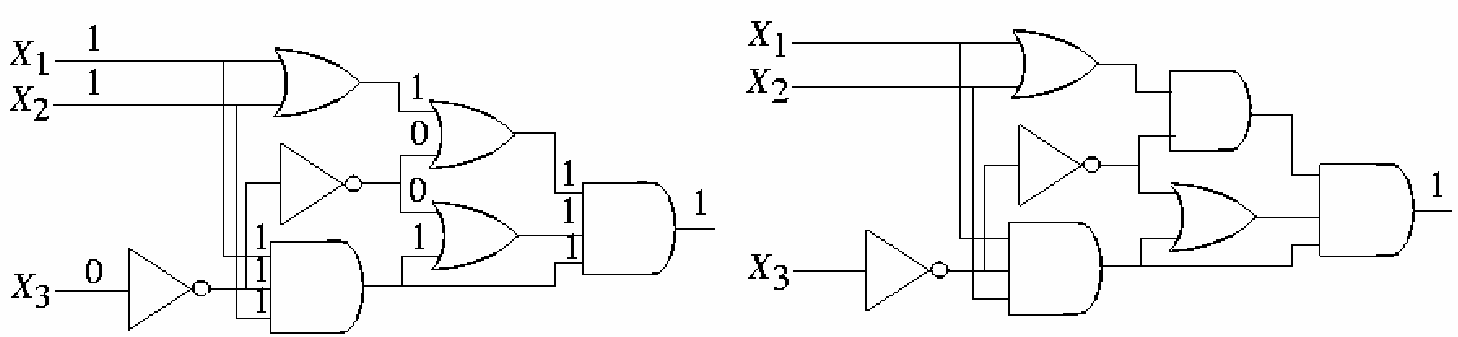


Decision Problem

- **Decision problems:** problem that can only be answered with "yes" or "no"
  - MST: Given a graph $G=(V, E)$ and a bound K, is there a spanning tree with a cost at most K?
  - TSP: Given a set of cities, distance between each pair of cities, and a bound B, is there a route that starts and ends at a given city, visits every city exactly once, and has total distance at most B?
- A decision problem Π, has instances: $I = (F, c, k)$
  - The set of of instances for which the answer is "yes" is given by Y_{Π}.
  - A subtask of a decision problem is *solution checking*: given $f \in F$, checking whether the cost is less than k.
- Could apply binary search on decision problems to obtain solutions to optimization problems.
- NP-completeness is associated with decision problems.

The Circuit-Satisfiability Problem (Circuit-SAT)

- **The Circuit-Satisfiability Problem (Circuit-SAT):**
  - **Instance:** A combinational circuit *C* composed of AND, OR, and NOT gates.
  - **Question:** Is there an assignment of Boolean values to the inputs that makes the output of *C* to be 1?
- A circuit is satisfiable if there exists a a set of Boolean input values that makes the output of the circuit to be 1.
  - Circuit (a) is satisfiable since $\langle x_1, x_2, x_3 \rangle = \langle 1, 1, 0 \rangle$ makes the output to be 1.



(a) satisfiable circuit (b) unsatisfiable circuit

Complexity Class P

- **Complexity class *P*** contains those problems that can be **solved** in polynomial time in the **size of input**.
  - **Input size:** size of encoded “binary” strings.
  - Edmonds: Problems in P are considered **tractable**.
- The computer concerned is a *deterministic Turing machine*
  - *Deterministic* means that each step in a computation is predictable.
  - A *Turing machine* is a mathematical model of a universal computer (any computation that needs polynomial time on a Turing machine can also be performed in polynomial time on any other machine).
- MST is in *P*.

Complexity Class NP

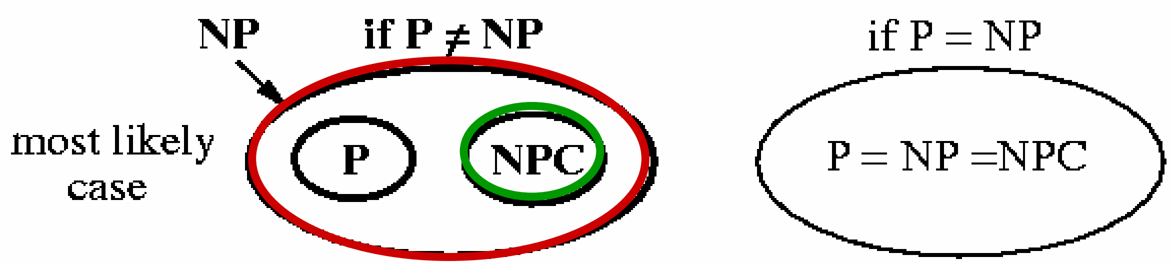
- Suppose that *solution checking* for some problem can be done in polynomial time on a deterministic machine $\Rightarrow$ the problem can be solved in polynomial time on a *nondeterministic Turing machine.*
  - *Nondeterministic*: the machine makes a guess, e.g., the right one (or the machine evaluates all possibilities in parallel).
- **The class NP** (**N**ondeterministic **P**olynomial): class of problems that can be **verified** in polynomial time in the size of input.
  - NP: class of problems that can be solved in polynomial time on a nondeterministic machine.
- Is TSP $\in$ NP?
  - Need to **check** a solution in polynomial time.
    - Guess a tour.
    - Check if the tour visits every city exactly once.
    - Check if the tour returns to the start.
    - Check if total distance $\leq B$.
  - All can be done in $O(n)$ time, so TSP $\in$ NP.

NP-Completeness

- An issue which is still unsettled:

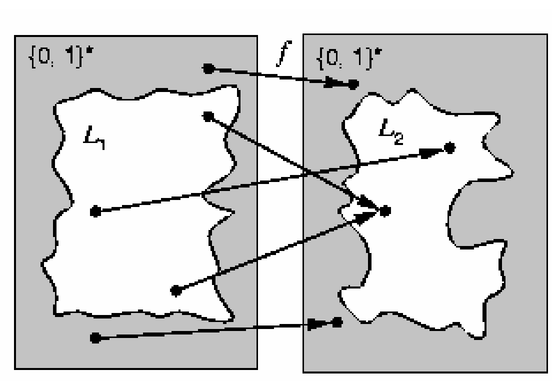
$$P \subset NP \text{ or } P = NP?$$

- There is a strong belief that $P \neq NP$, due to the existence of NP-complete problems.
- **The class NP-complete (NPC):**
  - Developed by S. Cook and R. Karp in early 1970.
  - All problems in NPC have the same degree of difficulty: **Any** NPC problem can be solved in polynomial time $\Rightarrow$ **all** problems in NP can be solved in polynomial time.

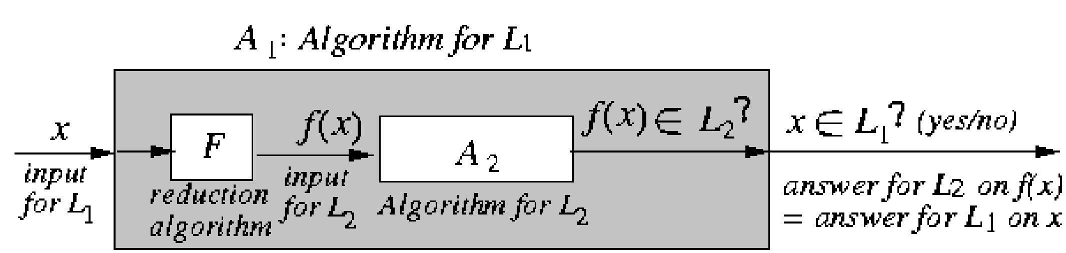


Polynomial-time Reduction

- **Motivation:** Let L_1 and L_2 be two decision problems. Suppose algorithm A_2 can solve L_2. Can we use A_2 to solve L_1?
- **Polynomial-time reduction *f* from L_1 to L_2: $L_1 \leq_P L_2$**
  - *f* reduces input for L_1 into an input for L_2 s.t. the reduced input is a "yes" input for L_2 iff the original input is a "yes" input for L_1.
    - $L_1 \leq_P L_2$ if $\exists$ polynomial-time computable function f: $\{0, 1\}^* \rightarrow \{0, 1\}^*$ s.t. $x \in L_1$ iff $f(x) \in L_2$, $\forall\, x \in \{0, 1\}^*$.
    - **L_2 is at least as hard as L_1.**
- *f* is computable in polynomial time.

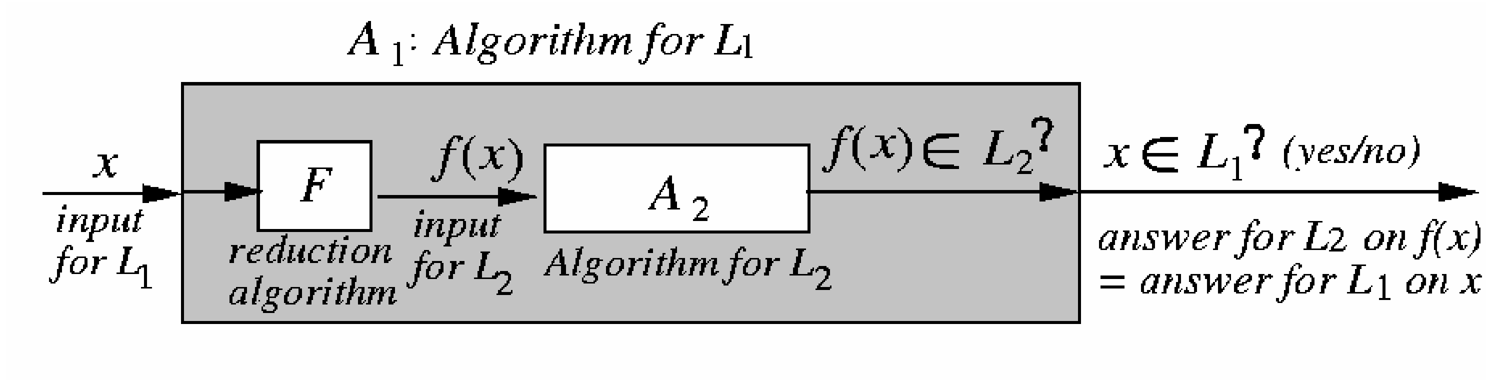


$L_1 \leq_p L_2$



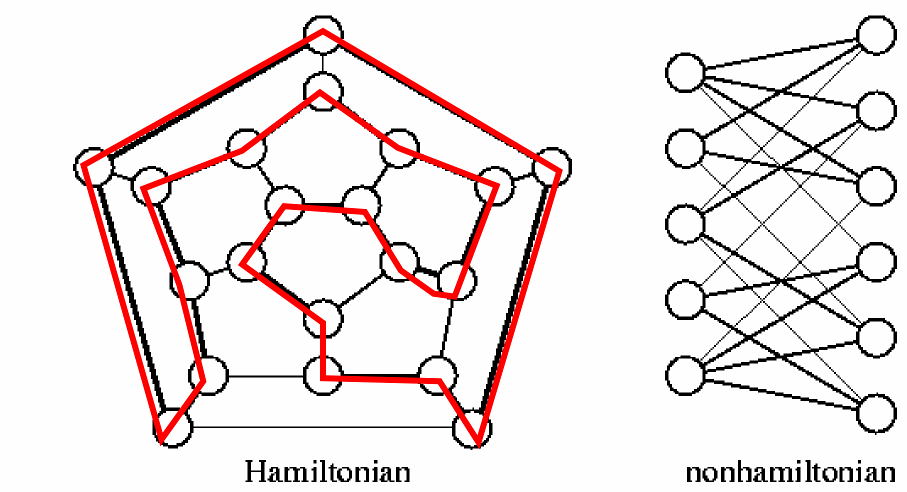
Significance of Reduction

- Significance of $L_1 \leq_P L_2$:
  - $\exists$ polynomial-time algorithm for $L_2 \Rightarrow \exists$ polynomial-time algorithm for L_1 ($L_2 \in \mathrm{P} \Rightarrow L_1 \in \mathrm{P}$).
  - $\nexists$ polynomial-time algorithm for $L_1 \Rightarrow \nexists$ polynomial-time algorithm for L_2 ($L_1 \notin \mathrm{P} \Rightarrow L_2 \notin \mathrm{P}$).
- $\leq_P$ is transitive, i.e., $L_1 \leq_P L_2$ and $L_2 \leq_P L_3 \Rightarrow L_1 \leq_P L_3$.



Polynomial-time Reduction

- The Hamiltonian Circuit Problem (HC)
  - **Instance:** an undirected graph $G = (V, E)$.
  - **Question:** is there a cycle in G that includes every vertex exactly once?
- TSP (The Traveling Salesman Problem)
- How to show HC $\leq_P$ TSP?
  1. Define a function f mapping any HC instance into a TSP instance, and show that f can be computed in polynomial time.
  2. Prove that G has an HC iff the reduced instance has a TSP tour with distance $\leq B$ ($x \in$ HC $\Leftrightarrow f(x) \in$ TSP).

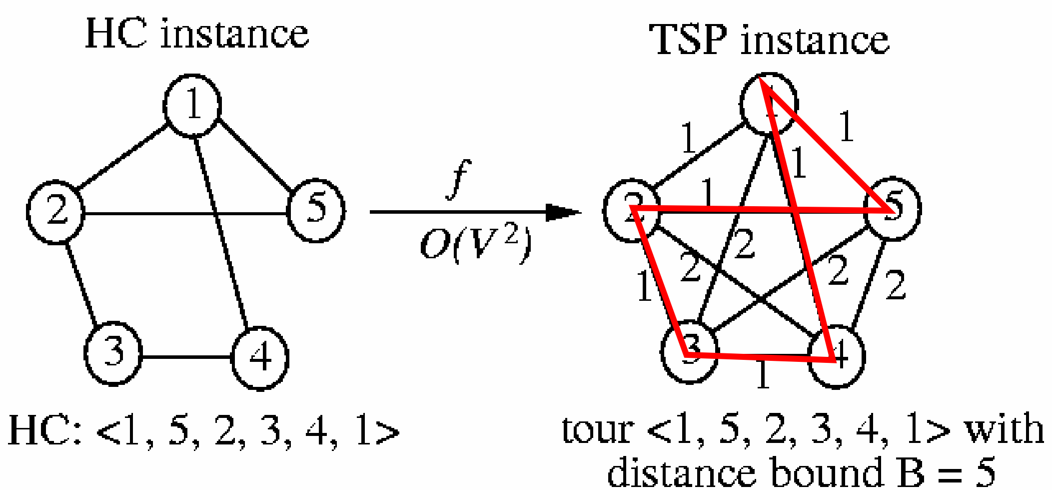


HC $\leq_P$ TSP: Step 1

1. Define a reduction function *f* for HC $\leq_P$ TSP.
   - Given an arbitrary HC instance $G = (V, E)$ with n vertices
     - Create a set of n cities labeled with names in V.
     - Assign distance between u and v

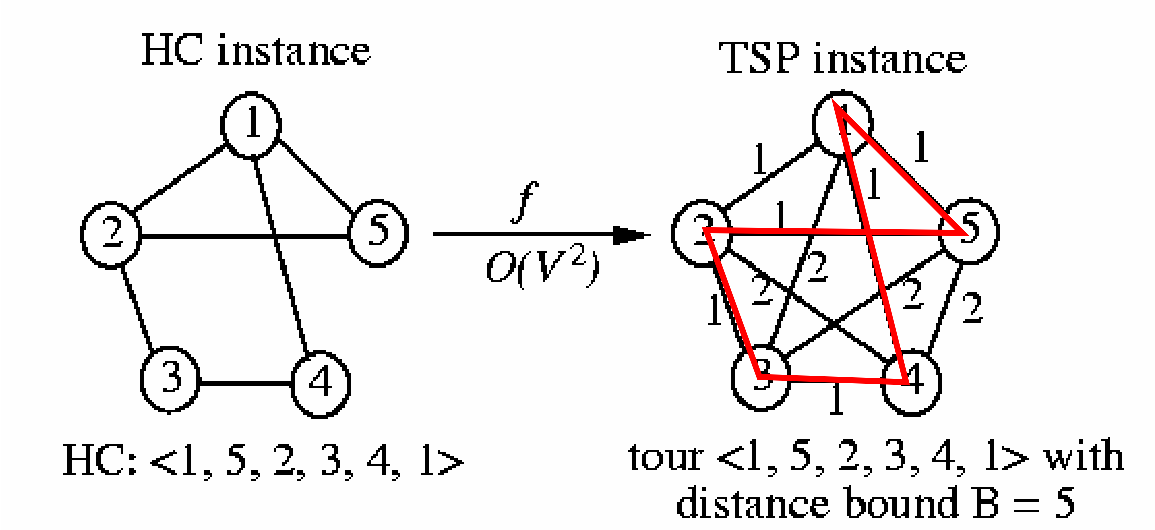
$$d(u,v) = \begin{cases} 1, & \text{if } (u,v) \in E, \\ 2, & \text{if } (u,v) \notin E. \end{cases}$$

     - Set bound $B = n$.
   - *f* can be computed in $O(V^2)$ time.



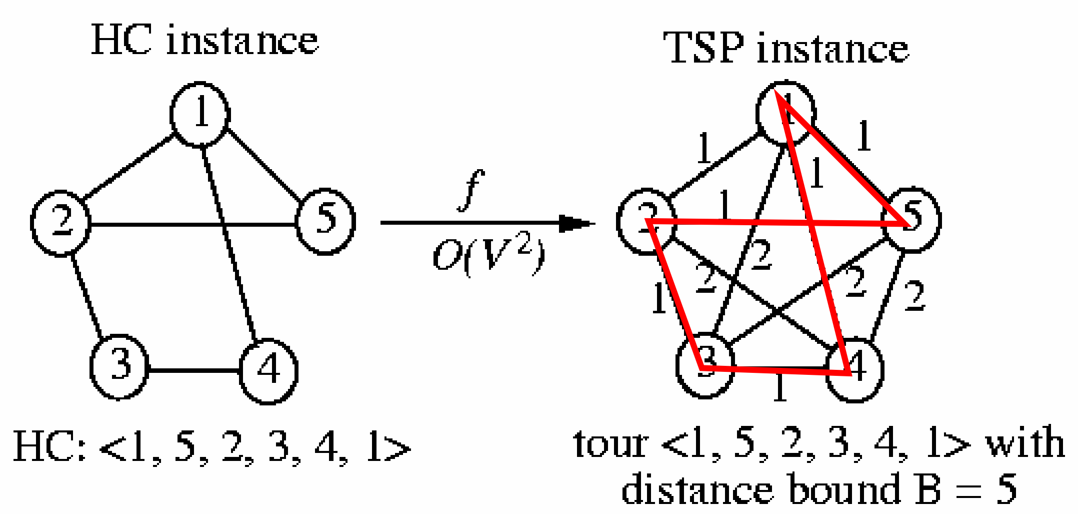
HC $\leq_P$ TSP: Step 2

2. *G* has an HC iff the reduced instance has a TSP with distance $\leq B$.
   - $x \in$ HC $\Rightarrow f(x) \in$ TSP.
     - Suppose the HC is $h = \langle v_1, v_2, \ldots, v_n, v_1 \rangle$. Then, *h* is also a tour in the transformed TSP instance.
     - The distance of the tour *h* is $n = B$ since there are *n* consecutive edges in *E*, and so has distance 1 in $f(x)$.
     - Thus, $f(x) \in$ TSP ($f(x)$ has a TSP tour with distance $\leq B$).



HC $\leq_P$ TSP: Step 2 (cont'd)

2. *G* has an HC iff the reduced instance has a TSP with distance $\leq B$.
   - $f(x) \in$ TSP $\Rightarrow x \in$ HC.
     - Suppose there is a TSP tour with distance $\leq n = B$. Let it be $<v_1, v_2, \ldots, v_n, v_1>$..
     - Since distance of the tour $\leq n$ and there are n edges in the TSP tour, the tour contains only edges in E.
     - Thus, $<v_1, v_2, \ldots, v_n, v_1>$ is a Hamiltonian cycle ($x \in$ HC).

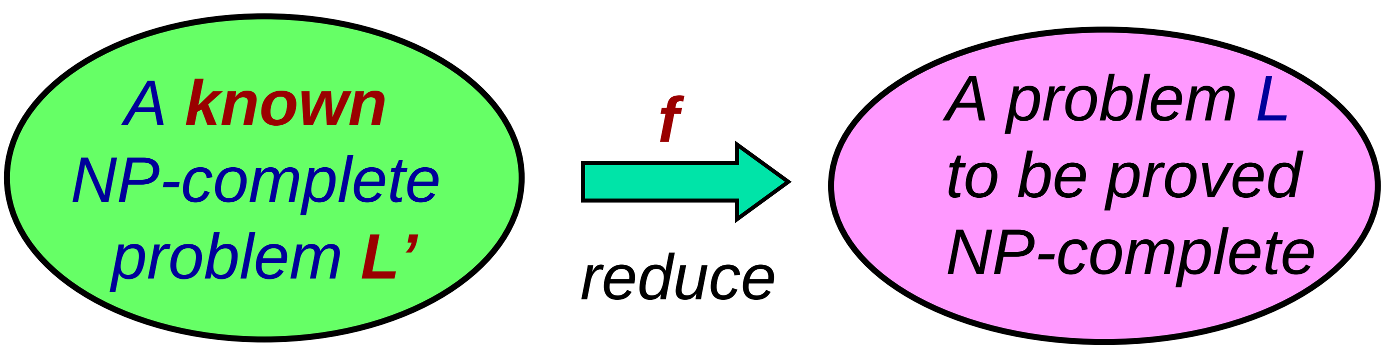


NP-Completeness and NP-Hardness

- **NP-completeness:** **worst-case** analyses for **decision** problems.
- L is **NP-complete** if
  - **L $\in$ NP**
  - **NP-Hard: L' $\leq_P$ L for every L' $\in$ NP.**
- **NP-hard:** If *L* satisfies the 2nd property, but not necessarily the 1st property, we say that *L* is **NP-hard**.
- Suppose $L \in$ NPC.
  - If $L \in P$, then there exists a polynomial-time algorithm for every $L' \in$ NP (i.e., P = NP).
  - If $L \notin P$, then there exists no polynomial-time algorithm for any $L' \in$ NPC (i.e., P $\neq$ NP).

Proving NP-Completeness

- **Five steps for proving that *L* is NP-complete:**
  1. Prove $L \in$ NP.
  2. Select a known NP-complete problem ***L'***.
  3. Construct a reduction *f* transforming **every** instance of ***L'*** to an instance of *L*.
  4. Prove that $x \in L'$ iff $f(x) \in L$ for all $x \in \{0, 1\}^*$.
  5. Prove that *f* is a polynomial-time transformation.
- We have shown that TSP is NP-complete.

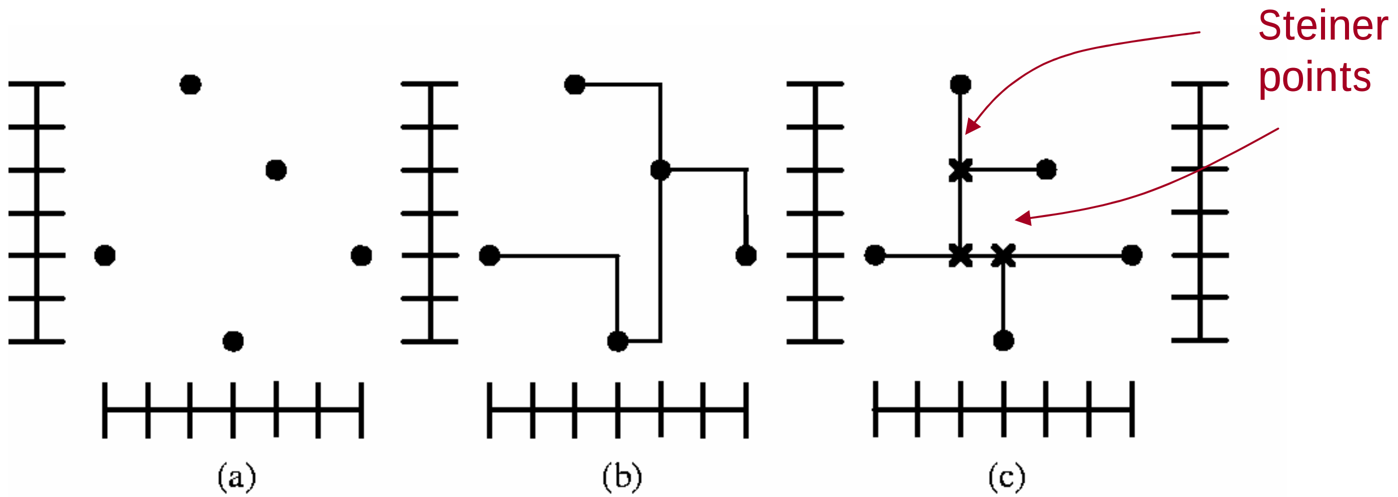


Coping with NP-hard problems

- **Approximation algorithms**
  - Guarantee to be a fixed percentage away from the optimum.
  - E.g., MST for the minimum Steiner tree problem.
- **Pseudo-polynomial time algorithms**
  - Has the form of a polynomial function for the complexity, but is not to the problem size.
  - E.g., $O(nW)$ for the 0-1 knapsack problem.
- **Restriction**
  - Work on some subset of the original problem.
  - E.g., the longest path problem in directed acyclic graphs.
- **Exhaustive search/Branch and bound**
  - Is feasible only when the problem size is small.
- **Local search:**
  - Simulated annealing (hill climbing), genetic algorithms, etc.
- **Heuristics:** No guarantee of performance.

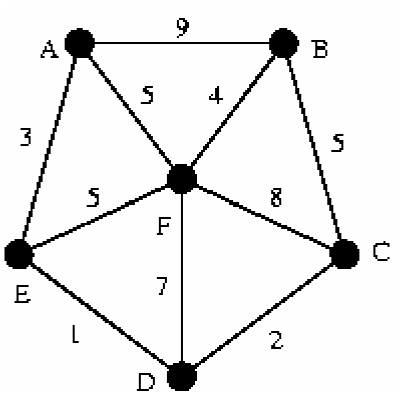
Spanning Tree v.s. Steiner Tree

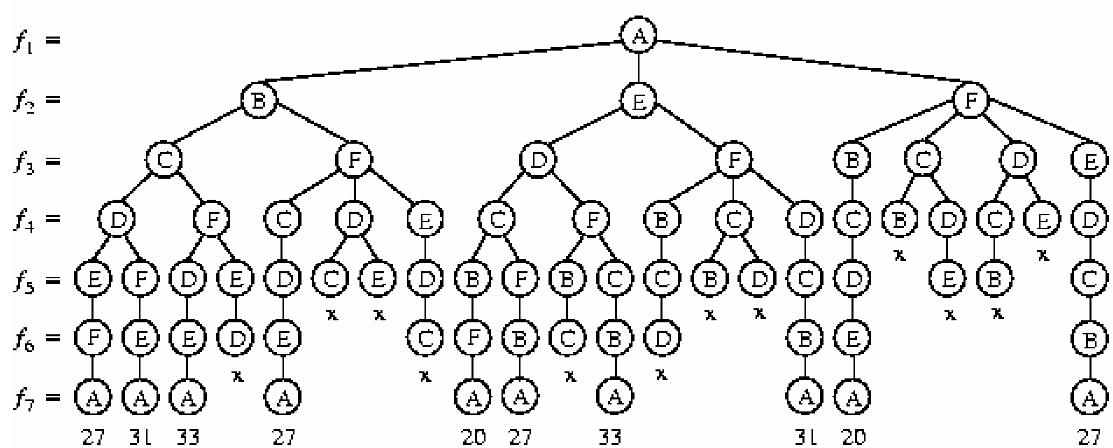
- **Manhattan distance:** If two points (nodes) are located at coordinates (x_1, y_1) and (x_2, y_2), the Manhattan distance between them is given by $d_{12} = |x_1-x_2| + |y_1-y_2|$.
- **Rectilinear spanning tree:** a spanning tree that connects its nodes using Manhattan paths (Fig. (b) below).
- **Steiner tree:** a tree that connects its nodes, and additional points (**Steiner points**) are permitted to used for the connections.
- The minimum rectilinear spanning tree problem is in P, while the minimum rectilinear Steiner tree (Fig. (c)) problem is NP-complete.
  - The spanning tree algorithm can be an *approximation* for the Steiner tree problem (at most 50% away from the optimum).



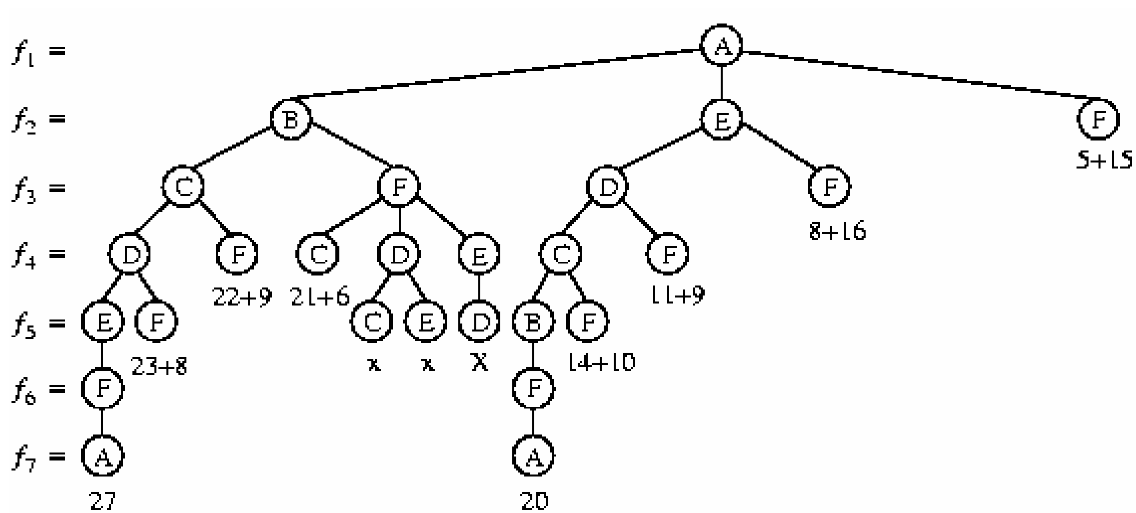
Exhaustive Search v.s. Branch and Bound

- TSP example





Backtracking/exhaustive search



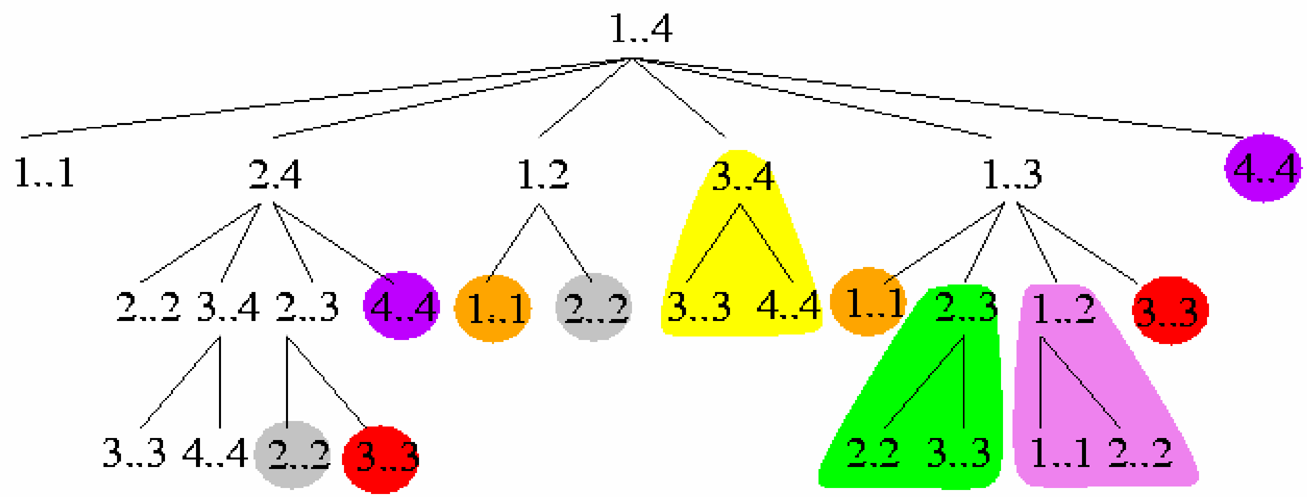
Branch and bound

Algorithmic Paradigms

- **Exhaustive search:** Search the entire solution space.
- **Branch and bound:** A search technique with pruning.
- **Greedy method:** Pick a locally optimal solution at each step.
- **Dynamic programming:** Partition a problem into a collection of sub-problems, the sub-problems are solved, and then the original problem is solved by combining the solutions. (Applicable when the sub-problems are **NOT independent**).
- **Hierarchical approach:** Divide-and-conquer.
- **Mathematical programming:** A system of solving an objective function under constraints.
- **Simulated annealing:** An adaptive, iterative, non-deterministic algorithm that allows "uphill" moves to escape from local optima.
- **Tabu search:** Similar to simulated annealing, but does not decrease the chance of "uphill" moves throughout the search.
- **Genetic algorithm:** A population of solutions is stored and allowed to evolve through successive generations via mutation, crossover, etc.

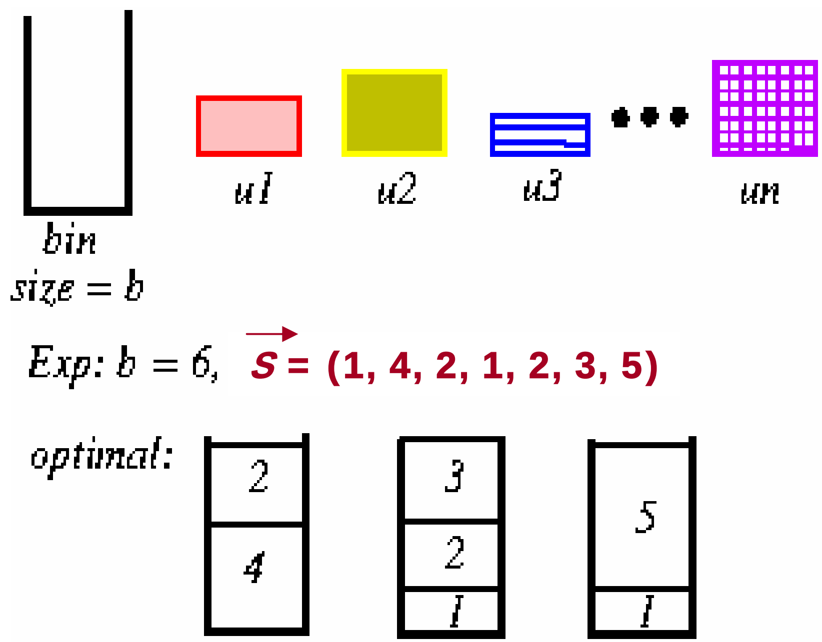
Dynamic Programming (DP) v.s. Divide-and-Conquer

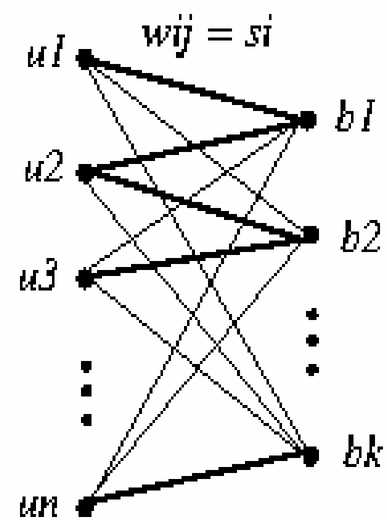
- Both solve problems by combining the solutions to subproblems.
- Divide-and-conquer algorithms
  - Partition a problem into **independent** subproblems, solve the subproblems recursively, and then combine their solutions to solve the original problem.
  - Inefficient if they solve the same subproblem more than once.
- Dynamic programming (DP)
  - Applicable when the subproblems are **not independent**.
  - DP solves each subproblem just once.



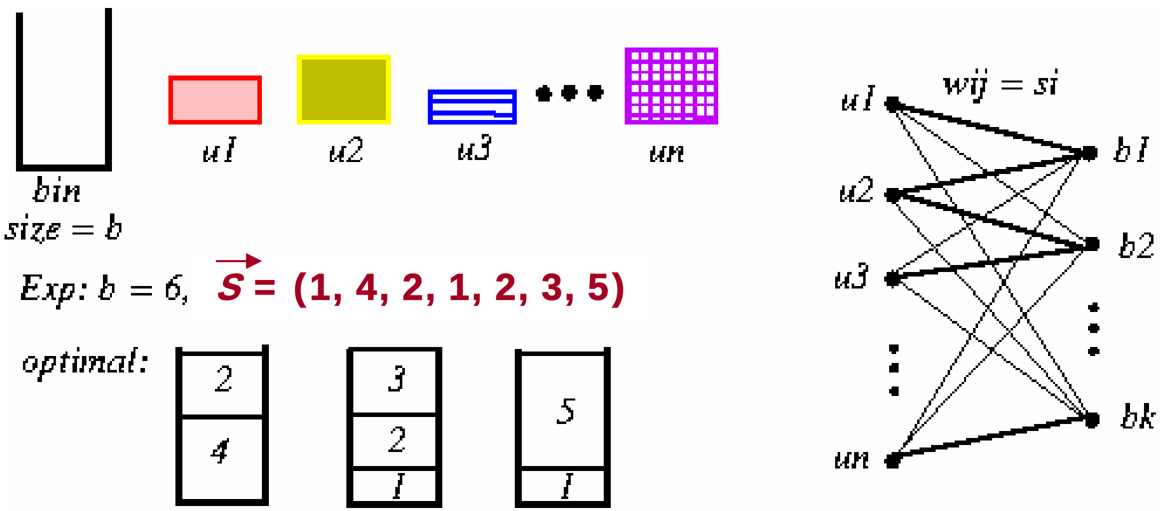
Example: Bin Packing

- **The Bin-Packing Problem** Π **:** Items $U = \{u_1, u_2, \ldots, u_n\}$, where u_i is of an integer size s_i; set B of bins, each with capacity b.
- **Goal:** Pack all items, minimizing # of bins used. (**NP-hard!**)





Algorithms for Bin Packing



- Greedy approximation alg.: First-Fit Decreasing (FFD)
  - $FFD(\Pi) \leq 11OPT(\Pi)/9 + 4$)
- Dynamic Programming? Hierarchical Approach? Genetic Algorithm? …
- Mathematical Programming: Use **integer linear programming (ILP)** to find a solution using |*B*| bins, then search for the smallest feasible |*B*|.

ILP Formulation for Bin Packing

- 0-1 variable: x_{ij}=1 if item u_i is placed in bin b_j, 0 otherwise.

$$
\begin{aligned}
\max \quad & \sum_{(i,j)\in E} w_{ij} x_{ij} \\
\textit{subject to} \quad & \\
& \sum_{\forall i \in U} w_{ij} x_{ij} \le b_j, \forall j \in B \quad /* \textit{ capacity constraint } */ && (1) \\
& \sum_{\forall j \in B} x_{ij} = 1, \forall i \in U \quad /* \textit{ assignment constraint } */ && (2) \\
& \sum_{ij} x_{ij} = n \quad /* \textit{ completeness constraint } */ && (3) \\
& x_{ij} \in \{0, 1\} \quad /* \; 0, \; 1 \textit{ constraint } */ && (4)
\end{aligned}
$$

- **Step 1:** Set |*B*| to the lower bound of the # of bins.
- **Step 2:** Use the ILP to find a feasible solution.
- **Step 3:** If the solution exists, the # of bins required is |*B*|. Then exit.
- **Step 4:** Otherwise, set |*B*| $\leftarrow$ |*B*| + 1. Goto Step 2.

CAD Related Conferences/Journals

- Important Conferences:
  - **ACM/IEEE Design Automation Conference (DAC)**
  - **IEEE/ACM Int'l Conference on Computer-Aided Design (ICCAD)**
  - IEEE Int'l Test Conference (ITC)
  - ACM Int'l Symposium on Physical Design (ISPD)
  - ACM/IEEE Asia and South Pacific Design Automation Conf. (ASP-DAC)
  - ACM/IEEE Design, Automation, and Test in Europe (DATE)
  - IEEE Int'l Conference on Computer Design (ICCD)
  - IEEE Custom Integrated Circuits Conference (CICC)
  - IEEE Int'l Symposium on Circuits and Systems (ISCAS)
  - Others: VLSI Design/CAD Symposium/Taiwan
- Important Journals:
  - **IEEE Transactions on Computer-Aided Design (TCAD)**
  - **ACM Transactions on Design Automation of Electronic Systems (TODAES)**
  - **IEEE Transactions on VLSI Systems (TVLSI)**
  - **IEEE Transactions on Computers (TC)**
  - IEE Proceedings – Circuits, Devices and Systems
  - IEE Proceedings – Digital Systems
  - INTEGRATION: The VLSI Journal